EXACT FINITE ELEMENT FORMULATION FOR COUPLED FLEXURAL-LATERAL-TORSIONAL-WARPING STATIC ANALYSIS OF THIN-WALLED ASYMMETRIC SHEAR DEFORMABLE MEMBERS

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الملخص

تم إشتقاق عنصرعارضة فائق التقارب (superconvergent beam element) بإستخدام نظرية العناصر المتناهية من أجل التحليل الإستاتيكي المقترن للإنحناء العرضي- الجانبي مع الإلتواء- الفتل للعارضات رقيقة الجدار ذات المقاطع المفتوحة الغير المتماثلة والمعرضة لأحمال متنوعة. تعتمد الصيغة الحالية على نظرية عارضة Timoshenko-Vlasov العامة حيث تتضمن بالكامل تأثيرات تشوه القص وفتل الإلتواء بالإضافة إلى الاقتران بين الإنحناء والإلتواء بسبب عدم تناسق المقطع العرضي. تم إشتقاق عائلة دوال الشكل الدقيقة (exact shape functions) بناءًعلى الحل الدقيق (closed form solution) لمعادلات الإتزان الإستاتيكية المقترنة. تم إستخدام دالة الشكل الدقيق التي تم إشتقاقها لصياغة مصفوفة الصلابة (stiffness matrix) ومتجه الحمل المكافئ للطاقة (potential load). عنصر العارضة (beam element) الذي تم إشتقاقه يحتوي على عقدتين وست درجات من الحرية لكل عقدة (six degrees of freedom per node)، وهو قادر على إلتقاط إقتران الإنحناء والإلتواء بشكل كامل. من أجل إثبات دقة وكفاءة العنصر، تم تقديم مقارنات مع حلول العناصر المحدودة الأخرى التي تم إنشاؤها باستخدام Abaqus. تبين النتائج أن عنصر العارضة المحسن بهذه الدراسة خاليًا من الأخطاء مقارنة بالاسلوب التقليدي للإشتقاق وينتج عنه إتفاق ممتاز (excellent agreement) مع حلول العناصر المحدودة الأخرى بفارق صغير في التكلفة الحسابية والنمذجة.

ABSTRACT

A super-convergent finite beam element is developed for the coupled flexural-lateral-torsional-warping static analysis of thin-walled beams with asymmetric open sections subjected to general loading. The present formulation is based on a generalized Timoshenko-Vlasov beam theory and fully captures the effects of shear deformation, the St. Venant and warping torsion as well as the coupling between bending and torsion due to the nonsymmetry of the cross-section. A family of exact shape functions is derived based on the closed form solution of the coupled equilibrium static equations. The exact shape functions developed are then employed to formulate the stiffness matrix and the energy equivalent load vector. The beam element developed has two-nodes and six degrees of freedom per node and is able to fully capture the flexural-torsional coupling. In order to demonstrate the exactness and efficiency of the element, comparisons are provided against other established finite element solutions under Abaqus. The element is shown to be free from discretization errors encountered under other interpolation schemes and yields results in excellent agreement with those based on other finite element solutions at a fraction of the computational and modeling cost.

KEYWORDS: Coupled Bending-Torsional-Warping Response; Static Analysis; Exact Shape Functions; Two-Nodded Beam Element.

INTRODUCTION AND OBJECTIVE

The conventional thin-walled beam theory developed by Vlasov [1] is extensively used for the static and dynamic analyses of members with open cross-sections. The theory is based on two kinematics assumptions: (1) the beam cross-section remains rigid in its own plane, and (2) the transverse shear deformations within the cross-section midsurface are considered negligible. The second assumption signifies that within the Vlasov theory, warping deformations are captured, whereas shear deformations are neglected. While Timoshenko [2] independently developed a similar theory for thin-walled open beams in which the transverse shear deformations are included. Although the Vlasov theory for thin-walled members of open cross-sections is well established, it presents limitations in the static and dynamic analyses of thin-walled open members. The range of applicability of the Vlasov's beam theory can be extended by taking into consideration shear deformation effects. Towards this goal, the present formulation dealing with coupled static analysis of thin-walled open members based on generalized Vlasov-Timoshenko beam theories are used.

The present survey focuses on the finite element formulations of thin-walled open members. Numerous finite element solutions for the analysis of thin-walled members subjected to general static and dynamic forces are based on two approaches. In the first approach, finite element formulations based on approximate polynomial interpolation functions are most common and include the work of Chen and Tamma [3], Hashemi and Richard [4,5], Lee and Kim [6,7], Lee and Lee [8], Kim and Kim [9], Voros [10,11], Vo and Lee [12,13] and Vo et al. [14,15], Kim [16] and Kim [17]. Among them, Chen and Tamma [3] used the finite element method in conjunction with an implicit-starting unconditionally stable methodology for the dynamic analysis of thin-walled open members under deterministic loads. Hashemi and Richard [4,5] developed a dynamic finite element for the coupled bending–torsional vibration analysis of thin-walled beams with/without axial loads effect. Their solution can be regarded as an intermediate method between the finite element method and the dynamic stiffness matrix method. The exact solutions of the governing dynamic equations of equilibrium were obtained and subsequently frequency-dependent hyperbolic interpolation functions were adopted to formulate the stiffness and mass matrices of the structure. By using linear and cubic Hermitian shape functions, Lee and Kim [6,7] investigated the coupled free vibration of thin-walled composite beams with doubly symmetric and channel-shaped cross-sections. Lee and Lee [8] developed a finite element model to investigate the flexural and torsional behavior of thin-walled composite I-beams with arbitrary laminate stacking sequence using a linear combination of the one-dimensional Lagrangian interpolation function for axial displacement and the Hermite-cubic interpolation function for lateral displacements and twist angle. In their formulation, the shear deformation effects were not considered. Kim and Kim [9] derived the coupled flexural-torsional free vibration of asymmetric thin-walled shear deformable beam using an isoparametric finite beam element. The influence of lateral forces on the coupled flexural-torsional free vibration of thin-walled open members was studied by Voros [10,11]. In his formulations, a two-noded beam element with fourteen degrees of freedom is formulated. Vo and Lee [12,13] and Vo et al. [14,15] studied the coupled flexural-torsional free vibration of thin-walled open composite beams under constant axial forces and end moments by developing a displacement-based one dimensional finite element model. Recently, Kim [16] developed a shear deformable beam element for the coupled flexural and torsional analyses of thin-walled composite I-beams with doubly and monosymmetric cross-sections. In his formulation, the isoparametric finite composite beam element based on the Lagrangian interpolation polynomials is presented. More recently, Kim [17] extended his work (Kim [16]) to investigate the coupled flexural-torsional static analysis for thin-walled channel composite beams resting on elastic foundations.

In the second approach, finite element formulations are based on the exact solution for dynamic and static equilibrium equations and they have two advantages: (i) they eliminate discretization errors encountered in conventional interpolation schemes and thus converge to the solution using a minimal number of degrees of freedom, and (ii) they lead to elements that are free from shear locking problems. Finite element solutions based on exact solutions for the dynamic equations of motion include the work of Hjaji and Mohareb [18,19]. Hjaji and Mohareb [18] developed a two-noded beam element for the flexural, lateral and torsional dynamic analyses of thin-walled open members with doubly symmetric section subjected to general harmonic forces. Their formulation was based on a generalized Timoshenko-Vlasov beam theory, in which the shear deformations effects due to bending and warping and translational and rotary inertias were incorporated. Recently, in Hjaji and Mohareb [19], a super-convergent finite beam element was formulated for steady state dynamic response of thin-walled doubly symmetric prismatic beams under harmonic excitations. The formulations were based on Vlasov beam theory and captured the St. Venant and warping deformation effects, and translational and rotary inertia. Formulations based on exact homogeneous solution for static equilibrium equations include the work of Mei [20], Hu et al. [21] and Hjaji and Mohareb [22]. Mei [20] developed a finite element for the coupled free vibration analysis of thin-walled beams which incorporated warping effects. Hu et al. [21] studied the coupled bending-torsional dynamic behavior of thin-walled beams of asymmetric cross-sections. Recently, in Hjaji and Mohareb [22], a finite element formulation is developed for the coupled flexural-torsional analysis of thin-walled open monosymmetric beams under general static forces. Based on a generalized Timoshenko-Vlasov beam theory, a two-noded finite element with four degrees of freedom per node was developed and fully captured the effects of warping stiffness, shear deformation, and the torsional-flexural coupling. However the solutions based on this approach are applicable only for doubly symmetric sections (Hjaji and Mohareb [18,19]) and mono-symmetric sections (Hjaji and Mohareb [22]) and thus are unable to capture the coupled flexural-lateral-torsional-warping static response in asymmetric sections. Hjaji and Mohareb [23] developed a superconvergent finite beam element for dynamic steady-state analysis of thin-walled members with asymmetric open cross-sections under harmonic forces. Their formulations were based on exact shape functions and captured the shear deformation effects caused by bending and warping, translational and rotary inertias, and bending-torsional coupling effects due to asymmetry of the cross-section. Hjaji and Mohareb [24] developed exact solutions for coupled flexural-lateral-torsional static response of thin-walled asymmetric open members subjected to general loading. The formulation was based on a generalized Timoshenko-Vlasov beam theory and accounted for the effects of shear deformation due to bending and warping, and captured the effects of flexural–torsional coupling due to cross-section asymmetry.

Within the above context and to the best knowledge of the authors of this paper, there is no publication reported on developing a finite element solution based on exact shape functions which satisfy the exact homogeneous solution of the coupled static equilibrium equations. Therefore, the present paper aims at developing a finite element formulation for coupled flexural-lateral-torsional-warping static response for thin-walled open asymmetric beams subjected to general static forces. The formulation sought is based on exact shape functions, and captures shear deformation effects due to bending and warping, and bending-torsional coupling effects due to the cross-section asymmetry.

MAIN ASSUMPTIONS

The present formulation is based on the following main assumptions:

1. The formulation is applicable to prismatic thin-walled members of arbitrary open cross-sections,
2. Cross-section is assumed to remain undeformed in its own plane but free to warp in the longitudinal direction (Vlasov assumption),
3. For the flexural response, the cross-section originally planar remains planar throughout deformation, but does not remain perpendicular to the centroidal axis after deformation, i.e., the transverse shear deformation of the mid-surface of the cross-section is incorporated in the assumed kinematics (Timoshenko beam assumption),
4. For the torsional response, the cross-section is assumed to undergo warping in a manner analogous to the Vlasov beam.
5. The deformations are assumed to remain linearly elastic during deformation, and
6. Strains and rotations are assumed small.

KINEMATICS RELATIONS

A thin-walled member of arbitrary open cross-section has a fixed right-handed orthogonal Cartesian coordinate system (X,Y,Z) with Z axis parallel to the longitudinal axis of the beam used to describe the geometry and displacements. Figure (1a) shows a local coordinate system (n,s,z) positioned on the contour (middle line of the cross-section) in which the coordinates n and s are measured along the normal and along the tangent to the middle surface in a contour at a point of interest. Based on the above assumptions, the in-plane displacements $u_p(z,s)$, $v_p(z,s)$ and longitudinal displacement $w_p(z,s)$ of a general point $p[x(s),y(s)]$ located on the mid-surface of the cross-section are respectively given by [24]:

$$v_p(z,s) = v(z) + \left[x(s) - x_s\right]\theta_z(z) \quad (1)$$

$$u_p(z,s) = u(z) - \left[y(s) - y_s\right]\theta_z(z) \quad (2)$$

$$w_p(z,s) = w(z) + y(s)\theta_x(z) - x(s)\theta_y(z) + +\omega(s)\psi(z) \quad (3)$$

in which $u(z)$ and $v(z)$ are the displacement components of the shear centre S_c along the principal directions X and Y, $w(z)$ is the average longitudinal displacement along the longitudinal axis Z, $\theta_x(z)$ and $\theta_y(z)$ are the rotations of the cross-section about X and Y principal axes, $\theta_z(z)$ is the rotation angle of the cross-section about the longitudinal axis, $\psi(z)$ is a function which characterizes the magnitude of the warping deformation, $\omega(s)$ is the warping function defined by $\omega(s) = \int_A h(s)dA$, $x(s)$ and $y(s)$ are the coordinates of point denoted by a curvilinear coordinates lying on the middle surface of the section, while x_s and y_s are the coordinate of the shear centre along the principal axes X and Y.

The in-plane displacements $u_p(z,s)$ and $v_p(z,s)$ of the general point $p(x,y)$ are resolved into tangential and normal displacement components $\xi(z,s)$ and $\eta(z,s)$ along the tangential s and normal n local coordinates, (Fig. 1b), yielding:

$$\xi(z,s) = u(z)cos\alpha + v(z)sin\alpha + h(z)\theta_z(z) \tag{4}$$

$$\eta(z,s) = v(z)cos\alpha - u(z)sin\alpha + r(z)\theta_z(z) \tag{5}$$

in which $h(s) = [x(s)-x_s]\sin\alpha + [y(s)-y_s]\cos\alpha$, $r(s) = [x(s)-x_s]\cos\alpha + [y(s)-y_s]\sin\alpha$, $\sin\alpha = dy(s)/ds$, $\cos\alpha = dx(s)/ds$, $h(s) = d\omega(s)/ds$, where $\alpha(s)$ is the angle between the tangent of the cross-section of point $p(x,y)$ and the X axis (Figure 1b), $h(s)$ is the perpendicular distance from the shear center S_c to the tangent to the contour at point $p(x,y)$ and $r(s)$ represents the magnitude of the perpendicular distance from the shear centre to the normal of the profile line at point $p(x,y)$ (see Figure. 1b).

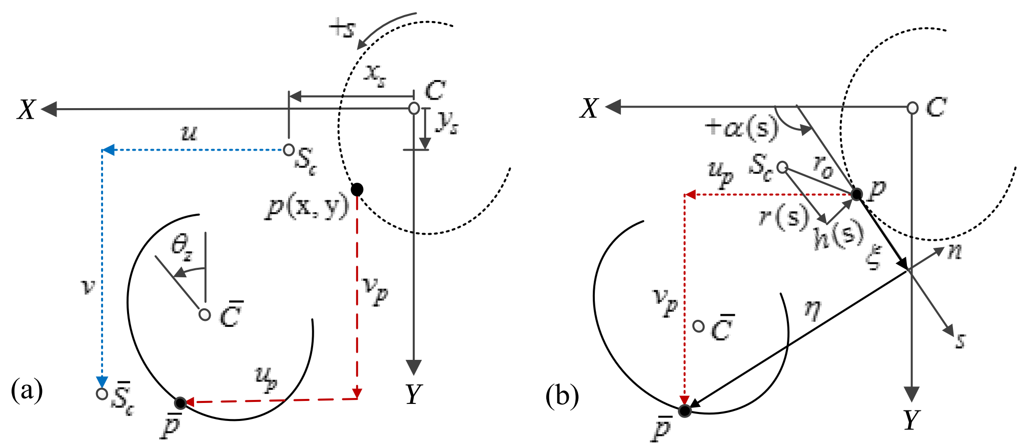


Figure 1: (a) Local coordinate system and displacement components of a point on the cross-section, and (b) tangential and normal displacements [8]

VARIATIONAL FORMULATION

The total potential energy Π of the thin-walled beam is defined as the sum of the internal strain energy U stored in the deformed body and the potential energy V due to applied loads, i.e., $\Pi = U + V$. Taking the first variation of Π and setting it equal to zero, one obtains:

$$\delta\Pi = \delta U + \delta V = 0 \tag{6}$$

in which, δU is the internal strain energy defined by [24]:

$$\delta U = \int_0^L \int_A [E\varepsilon_{zz}\delta\varepsilon_{zz} + G\gamma_{zs}\delta\gamma_{zs}] dAdz + \int_0^L GJ\theta_z'\delta\theta_z' dz \tag{7}$$

where, E is the modulus of elasticity, G is the shear modulus, J is the St. Venant torsional constant, A is the cross-sectional area, $\varepsilon_{zz} = \partial w_p/\partial z$ is the longitudinal strain while the shear strain is $\gamma_{zs} = \partial w_p/\partial s + \partial\xi/\partial z$. All primes denote derivatives with respect to coordinate z . The potential of the applied forces δV is given by [24]:

$$\begin{aligned}\delta V = &\int_0^L \left[q_z\delta w + q_x\delta u + q_y\delta v + m_z\delta\theta_z + m_x\delta\theta_x + m_y\delta\theta_y + m_w\delta\psi\right]dz \\ &+ [N_z\delta w]_0^L + [V_x\delta u]_0^L + \left[V_y\delta v\right]_0^L + [M_x\delta\theta_x]_0^L + \left[M_y\delta\theta_y\right]_0^L + [M_z\delta\theta_z]_0^L + [M_w\delta\psi]_0^L\end{aligned} \tag{8}$$

where $q_z(z), q_x(z)$ and $q_y(z)$ are the distributed longitudinal, transverse and lateral forces, $m_x(z)$, $m_y(z)$, $m_z(z)$ and $m_w(z)$ are the distributed bending and twisting moments and bimoment, $N_z(z_e)$, $V_x(z_e)$ and $V_y(z_e)$ are the concentrated longitudinal, transverse and lateral forces, $M_z(z_e), M_x(z_e), M_y(z_e)$ are the end moments and $M_w(z_e)$ is the end bimoments. All forces and moments applied at beam ends $(z_e = 0, L)$. All applied forces are assumed to have the same sign convention as those of the end displacements (Figure 1).

EQUILIBRIUM GOVERNING FIELD EQUATIONS

From equations (1-4), by substituting into energy equations (7) and (8). The resulting energy equations are substituted into equation (6) and by enforcing the orthogonally conditions; $\left\langle \int_A [x, y, xy, x\omega, y\omega, \omega] dA \right\rangle = \langle 0 \rangle$ and performing integration by parts with respect to coordinate z, the governing equilibrium equations are then obtained [8] as:

$$EAw'' = -q_z(z) \tag{9}$$

$$G\left[D_{xx}\left(u''-\theta_y'\right)+D_{xy}\left(v''+\theta_x'\right)+D_{hx}\left(\theta_z''+\psi'\right)\right]=-q_x(z) \tag{10}$$

$$G\left[D_{xy}\left(u''-\theta_y'\right)+D_{yy}\left(v''+\theta_x'\right)+D_{hy}\left(\theta_z''+\psi'\right)\right]=-q_y(z) \tag{11}$$

$$-EI_{xx}\theta_x''+G\left[D_{xy}\left(u'-\theta_y\right)+D_{yy}\left(v'+\theta_x\right)+D_{hy}\left(\theta_z'+\psi\right)\right]=m_x(z) \tag{12}$$

$$-EI_{yy}\theta_y''-G\left[D_{xx}\left(u'-\theta_y\right)+D_{xy}\left(v'+\theta_x\right)+D_{hx}\left(\theta_z'+\psi\right)\right]=m_y(z) \tag{13}$$

$$GJ\theta_z''+G\left[D_{hx}\left(u''-\theta_y'\right)+D_{hy}\left(v''+\theta_x'\right)+D_{\omega\omega}\left(\theta_z''+\psi'\right)\right]=-m_z(z) \tag{14}$$

$$-EC_w\psi''+G\left[D_{hx}\left(u'-\theta_y\right)+D_{hy}\left(v'+\theta_x\right)+D_{\omega\omega}\left(\theta_z'+\psi\right)\right]=-m_w(z) \tag{15}$$

The related boundary conditions are obtained as:

$$\left[EAw'-N_z(z)\right]\delta w(z)\Big|_0^L=0 \tag{16}$$

$$\left[GD_{xx}\left(u'-\theta_y\right)+GD_{xy}\left(v'+\theta_x\right)+GD_{hx}\left(\theta_z'+\psi\right)-V_x(z)\right]\delta u(z)\Big|_0^L=0 \tag{17}$$

$$\left[GD_{xy}\left(u'-\theta_y\right)+GD_{yy}\left(v'+\theta_x\right)+GD_{hy}\left(\theta_z'+\psi\right)-V_y(z)\right]\delta v(z)\Big|_0^L=0 \tag{18}$$

$$\left[EI_{xx}\theta_x'+M_x(z)\right]\delta\theta_x(z)\Big|_0^L=0 \tag{19}$$

$$\left[EI_{yy}\theta_y'-M_y(z)\right]\delta\theta_y(z)\Big|_0^L=0 \tag{20}$$

$$\left[GJ\theta_z'+GD_{hx}\left(u'-\theta_y\right)+GD_{hy}\left(v'+\theta_x\right)+GD_{\omega\omega}\left(\theta_z'+\psi\right)-M_z(z)\right]\delta\theta_z(z)\Big|_0^L=0 \tag{21}$$

$$\left[EC_w\psi'+M_w(z)\right]\delta\psi(z)\Big|_0^L=0 \tag{22}$$

In the above Equations, the cross-sectional properties are defined as:

$$A, I_{xx}, I_{yy}, D_{xx}, D_{yy}, D_{xy}, D_{hx}, D_{hy}, D_{ww}$$
$$= \int_A \left[1, y^2, x^2, \left(\frac{dx}{ds}\right)^2, \left(\frac{dy}{ds}\right)^2, \left(\frac{dy}{ds}\right)\left(\frac{dy}{ds}\right), \left(\frac{d\omega}{ds}\right)\left(\frac{dx}{ds}\right), \left(\frac{d\omega}{ds}\right)\left(\frac{dy}{ds}\right), \left(\frac{d\omega}{ds}\right)^2 \right] dA \tag{23}$$

Equation (9) provides the longitudinal vibration response of the beam which is uncoupled from the remaining coupled field equations and can be solved independently [e.g., Hjaji and Mohareb 2011]. Equations (10-15) and associated boundary conditions (17-22) govern the coupled biaxial bending-torsional-warping static response. The analytical closed-form static analysis of the coupled field equations presented in (10-15) was previously investigated in [8]. The present work is focused only on the finite element formulation for the coupled system of static field equations (10-15).

HOMOGENEOUS SOLUTION FOR COUPLED STATIC EQUATIONS

The exact homogeneous solution of the governing coupled static equations (10-15) is obtained by setting the loading terms in the coupled field equations to zero, i.e., $q_x(z)=q_y(z)=m_x(z)=m_y(z)=m_z(z)=m_w(z)=0$. The solution of the displacement functions is assumed to take the following exponential form:

$$\langle \bar{W}(z) \rangle_{1\times6} = \langle u(z) \;\; v(z) \;\; \theta_x(z) \;\; \theta_y(z) \;\; \theta_z(z) \;\; \psi(z) \rangle_{1\times6} = \langle c_i \rangle_{1\times6} e^{m_i z}, \text{ for } i=1,2,3,....,6 \tag{24}$$

Where $\langle \bar{W}(z) \rangle_{1\times6} = \langle u(z) \;\; v(z) \;\; \theta_x(z) \;\; \theta_y(z) \;\; \theta_z(z) \;\; \psi(z) \rangle_{1\times6}$ is the vector of transverse, lateral, torsional and warping deformation functions, and $\langle c_i \rangle_{1\times6} = \langle c_1 \;\; c_2 \;\; c_3 \;\; c_4 \;\; c_5 \;\; c_6 \rangle_{1\times6}$ is the vector of unknown integration constants. From the displacement functions in equation (24), by substituting into coupled equations (10-15), rewritten in matrix form yielding:

$$\begin{bmatrix} GD_{xx}m_i^2 & GD_{xy}m_i^2 & GD_{xy}m_i & -GD_{xx}m_i & GD_{hx}m_i^2 & GD_{hx}m_i \\ & GD_{yy}m_i^2 & GD_{yy}m_i & -GD_{xy}m_i & GD_{hy}m_i^2 & GD_{hy}m_i \\ & & GD_{yy}-EI_{xx}m_i^2 & -GD_{xy} & GD_{hy}\mathcal{D} & GD_{hy} \\ & & & GD_{xx}-EI_{yy}m_i^2 & -GD_{hx}m_i & -GD_{hx} \\ & Symm & & & G(D_{\omega\omega}+J)m_i^2 & GD_{\omega\omega}m_i \\ & & & & & GD_{\omega\omega}-EC_w m_i^2 \end{bmatrix}_{i,6\times6}$$
$$\times \begin{bmatrix} e^{m_1 z} & 0 & 0 & 0 & 0 & 0 \\ & e^{m_2 z} & 0 & 0 & 0 & 0 \\ & & e^{m_3 z} & 0 & 0 & 0 \\ & & & e^{m_4 z} & 0 & 0 \\ & Symm & & & e^{m_4 z} & 0 \\ & & & & & e^{m_4 z} \end{bmatrix}_{i,6\times6} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix}_{i,6\times1} = \{0\}_{6\times1} \tag{25}$$

Where $r_o^2 = (1/A)\int_A \left(h^2 + r^2\right) dA = x_s^2 + y_s^2 + \left(I_{xx} + I_{yy}\right)/A$ is the polar radius of gyration about the shear centre. For a non-trivial solution, the determinant of the matrix in equation (25) is set to vanish leading to the quadratic eigenvalue problem equation of the form:

$$\left(m_i^2\left[\widehat{M}\right]_{6\times6}+m_i\left[\widehat{C}\right]_{6\times6}+\left[\widehat{K}\right]_{6\times6}\right)\{c\}_{i,6\times1}=0 \tag{26}$$

Where $\{c\}_i$ are the eigenvectors corresponding to eigenvalues m_i, and matrices $\left[\widehat{M}\right]_{6\times6}$, $\left[\widehat{C}\right]_{6\times6}$ and $\left[\widehat{K}\right]_{6\times6}$ are defined by:

$$\left[\widehat{M}\right]_{6\times6}=\begin{bmatrix} GD_{xx} & GD_{xy} & 0 & 0 & GD_{hx} & 0 \\ & GD_{yy} & 0 & 0 & GD_{hy} & 0 \\ & & -EI_{xx} & 0 & 0 & 0 \\ & & & -EI_{yy} & 0 & 0 \\ & Sym. & & & G\left(D_{\omega\omega}+J\right) & 0 \\ & & & & & -EC_w \end{bmatrix}_{6\times6},$$

$$\left[\widehat{C}\right]_{6\times6}=\begin{bmatrix} 0 & 0 & GD_{xy} & -GD_{xx} & 0 & GD_{hx} \\ & 0 & GD_{yy} & -GD_{xy} & 0 & GD_{hy} \\ & & 0 & 0 & GD_{hy} & 0 \\ & & & 0 & -GD_{hx} & 0 \\ & Sym. & & & 0 & GD_{\omega\omega} \\ & & & & & 0 \end{bmatrix}_{6\times6}, \text{ and}$$

$$\left[\widehat{K}\right]_{6\times6}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & GD_{yy} & -GD_{xy} & 0 & GD_{hy} \\ & & & GD_{xx} & 0 & -GD_{hx} \\ & Sym. & & & 0 & 0 \\ & & & & & GD_{\omega\omega} \end{bmatrix}_{6\times6}.$$

The quadratic 6×6 eigenvalue problem defined in Equation (26) is transformed into an equivalent 12×12 unsymmetrical linear eigenvalue problem as:

$$\left(m_i\begin{bmatrix}[I_6]_{6\times6} & [0]_{6\times6} \\ [0]_{6\times6} & \left[\widehat{M}\right]_{6\times6}\end{bmatrix}_{12\times12}-\begin{bmatrix}[0]_{6\times6} & [I_6]_{6\times6} \\ \left[-\bar{K}\right]_{6\times6} & \left[-\bar{C}\right]_{6\times6}\end{bmatrix}_{12\times12}\right)\begin{Bmatrix}\{c\}_{i,6\times6} \\ m_i\{c\}_{i,6\times6}\end{Bmatrix}_{12\times1}=\{0\}_{12\times1} \tag{27}$$

In which $[I_6]_{6\times6}$ is the 6×6 identity matrix. The non-trivial solution of equation (27) is given by the right Eigen-value arising by setting the determinant of the matrix in (27) to vanish. The generalized eigenvalues and corresponding eigenvectors are then determined numerically. For thin-walled beams with asymmetric section, it is observed that all twelve roots are non-zero and distinct (i.e., $m_i \neq m_j$ for $i \neq j$). Thus, the homogeneous solution of system of coupled equations (10-15) takes the form:

$$\{\bar{W}(z)\}_{6\times1} = [\bar{G}]_{6\times12}[\bar{E}(z)]_{12\times12}\{C_i\}_{12\times1} \tag{28}$$

Where $[\bar{E}(z)]_{12\times12} = diag\left[e^{m_1 z} \mid e^{m_2 z} \mid e^{m_3 z} \mid \; .... \mid e^{m_8 z}\right]_{12\times12}$ is a diagonal matrix of the exponential functions matrix $e^{m_i z}$ (m_i being the eigenvalues of corresponding eigenvalues), $[\bar{G}]_{6\times12}$ contains the eigenvectors of the quadratic eigenvalue problem arising from the coupled system of flexural-lateral-torsional equations, and $\{C_i\}_{12\times1}$ is a vector of unknown integration constants.

FINITE ELEMENT FORMULATION

This section formulates a two-nodded finite beam element with six degrees of freedom per node. The beam element is based on a family of shape functions, which exactly satisfy the homogeneous solution of the coupled static equations.

Formulation of Exact Shape Functions

In the present study, the vector of constants $\{C_i\}_{12\times1}$ can be expressed in terms of the nodal displacements by enforcing the conditions: $u(0)=u_1$, $v(0)=v_1$, $\theta_x(0)=\theta_{x1}$, $\theta_y(0)=\theta_{y1}$, $\theta_z(0)=\theta_{z1}$, $\psi(0)=\psi_1$ and $u(\ell)=u_2$, $v(\ell)=v_2$, $\theta_x(\ell)=\theta_{x2}$, $\theta_y(\ell)=\theta_{y2}$, $\theta_z(\ell)=\theta_{z2}$, $\psi(\ell)=\psi_2$. The displacement functions $\langle \bar{W}(z)\rangle_{1\times6} = \langle u(z)\ v(z)\ \theta_x(z)\ \theta_y(z)\ \theta_z(z)\ \psi(z)\rangle_{1\times6}$ are expressed in terms of nodal displacements $\langle d_N\rangle_{1\times12} = \langle u_1 \mid v_1 \mid \theta_{x1} \mid \theta_{y1} \mid \theta_{z1} \mid \psi_1 \mid u_2 \mid v_2 \mid \theta_{x2} \mid \theta_{y2} \mid \theta_{z2} \mid \psi_2\rangle_{1\times12}$, yielding:

$$\{d_N(z)\}_{12\times1} = \begin{Bmatrix} \{\bar{W}(0)\}_{6\times1} \\ \hdashline \{\bar{W}(\ell)\}_{6\times1} \end{Bmatrix}_{12\times1} = \begin{bmatrix} [\bar{G}]_{6\times12}[\bar{E}(0)]_{12\times12} \\ \hdashline [\bar{G}]_{6\times12}[\bar{E}(\ell)]_{12\times12} \end{bmatrix}_{12\times12} = [\Phi]_{12\times12}\{C_i\}_{12\times1} \tag{29}$$

From Equation (29), by substituting into Equation (28), one obtains:

$$\{\bar{W}(z)\}_{6\times1} = [\bar{G}]_{6\times12}[\bar{E}(z)]_{12\times12}[\Phi]^{-1}_{12\times12}\{d_N\}_{12\times1} = [H(z)]_{6\times12}\{d_N\}_{12\times1} \tag{30}$$

Where

$$[H(z)]_{6\times12} = [\bar{G}]_{6\times12}[\bar{E}(z)]_{12\times12}[\Phi]^{-1}_{12\times12}$$
$$= \left[\{H_{1,j}(z)\}_{12\times1} \mid \{H_{2,j}(z)\}_{12\times1} \mid \{H_{3,j}(z)\}_{12\times1} \mid \{H_{4,j}(z)\}_{12\times1} \mid \{H_{5,j}(z)\}_{12\times1} \mid \{H_{6,j}(z)\}_{12\times1}\right]_{6\times12}$$

the matrix of the shape functions which exactly satisfy the homogeneous form of the coupled equilibrium static equations and are dependent on the beam span and cross-sectional geometry.

Matrix Formulation

The variation of internal strain energy given by equation (7) is obtained in terms of nodal degrees of freedom as:

$$\delta U = \int_0^{\ell}\left[\langle \delta\bar{W}'(z)\rangle_{1\times6}[Y_a]_{6\times6}\{\bar{W}'(z)\}_{6\times1} + \langle \delta\bar{W}_d(z)\rangle_{1\times6}[Y_d]_{6\times6}\{\bar{W}_d(z)\}_{6\times1}\right]dz \tag{31}$$

Where $\langle \bar{W}'(z)\rangle_{1\times 6} = \langle u'(z) \mid v'(z) \mid \theta'_x(z) \mid \theta'_y(z) \mid \theta'_z(z) \mid \psi'(z)\rangle_{1\times 6}$,
$\langle \bar{W}_d(z)\rangle_{1\times 6} = \langle u'(z) \mid v'(z) \mid \theta_x(z) \mid \theta_y(z) \mid \theta'_z(z) \mid \psi(z)\rangle_{1\times 6}$,

$$[Y_d]_{6\times 6} = \begin{bmatrix} GD_{xx} & GD_{xy} & GD_{xy} & -GD_{xx} & GD_{hx} & GD_{hx} \\ & GD_{yy} & GD_{yy} & -GD_{xy} & GD_{hy} & GD_{hy} \\ & & GD_{yy} & -GD_{xy} & GD_{hy} & GD_{hy} \\ & & & GD_{xx} & -GD_{hx} & -GD_{hx} \\ & Symm & & & GD_{\omega\omega} & GD_{\omega\omega} \\ & & & & & GD_{\omega\omega} \end{bmatrix}_{6\times 6}$$, and

$[Y_a]_{6\times 6} = diag\left[0 \mid 0 \mid EI_{xx} \mid EI_{yy} \mid GJ \mid EC_w\right]_{6\times 6}$.

The first variation of the potential energy δV of the applied static forces is given by:

$$\delta V = -\left[\langle \delta\bar{W}(z)\rangle_{1\times 6}\{F_d(z)\}_{6\times 1}\right]_0^{\ell} - \int_0^{\ell}\langle \delta\bar{W}(z)\rangle_{1\times 6}\{F_c\}_{6\times 1}\,dz \tag{32}$$

Where $\langle F_c\rangle_{1\times 6} = \left\langle V_x(z)\big|_0^{\ell} \;\; V_y(z)\big|_0^{\ell} \;\; M_x(z)\big|_0^{\ell} \;\; M_y(z)\big|_0^{\ell} \;\; M_z(z)\big|_0^{\ell} \;\; M_w(z)\big|_0^{\ell}\right\rangle_{1\times 6}$.

From equations (31) and (32), by substituting into variational form represented by equation (6), one recovers:

$$\int_0^{\ell}\Big[\langle \delta\bar{W}'(z)\rangle_{1\times 6}[Y_a]_{6\times 6}\{\bar{W}'(z)\}_{6\times 1} + \langle \delta\bar{W}_d(z)\rangle_{1\times 6}[Y_d]_{6\times 6}\{\bar{W}_d(z)\}_{6\times 1}$$
$$-\langle \delta\bar{W}(z)\rangle_{1\times 6}\{F_c\}_{6\times 1}\Big]dz - \left[\langle \delta\bar{W}(z)\rangle_{1\times 6}\{F_d(z)\}_{6\times 1}\right]_0^{\ell} = 0 \tag{33}$$

From equation (30), by substituting into equation (33), one obtains:

$$[K_e]_{12\times 12}\{d_N\}_{12\times 1} = \{F_e\}_{12\times 1} \tag{34}$$

in which $[K_e]_{12\times 12} = \int_0^{\ell}\Big[[H'(z)]^T_{12\times 6}[Y_a]_{6\times 6}[H'(z)]_{6\times 12} + [H_d(z)]^T_{12\times 6}[Y_d]_{6\times 6}[H_d(z)]_{6\times 12}\Big]dz$ is the element stiffness matrix, and $\{F_e\}_{12\times 1} = \int_0^{\ell}\Big[[H(z)]^T_{12\times 6}\{F_d\}_{6\times 1}\Big]dz + \Big[[H(z)]^T_{12\times 6}\{F_c(z)\}_{6\times 1}\Big]_0^{\ell}$ is the element load vector, where

$[H_d(z)]_{6\times 12} = \Big[\{H'_{1,j}(z)\}_{12\times 1} \mid \{H_{2,j}(z)\}_{12\times 1} \mid \{H'_{3,j}(z)\}_{12\times 1} \mid \{H_{4,j}(z)\}_{12\times 1} \mid \{H'_{5,j}(z)\}_{12\times 1} \mid \{H_{6,j}(z)\}_{12\times 1}\Big]_{6\times 12}$.

NUMERICAL EXAMPLES AND DISCUSSION

In this section, two examples for thin-walled open beams of asymmetric cross-sections subjected to general static forces and various boundary conditions are presented to assess the validity, exactness and applicability of the present finite element formulation (using a single two-nodded beam element having twelve degrees of freedom). The finite element formulation developed in the present paper is based on the exact shape functions which exactly satisfy the homogeneous solution of the coupled static equilibrium equations derived in this paper [24]. Due to this treatment, the mesh discretization errors induced in the finite element formulations using polynomial shape functions are eliminated. As a result, it is observed that, the results obtained based on a single finite beam element exactly matched with those based on the exact closed-form solutions up to five significant digits. For the sake of comparison, three solutions

(Vlasov exact solution and two finite element solutions based on Abaqus shell and beam models) are provided as:

(1) Exact solution based on the Vlasov beam theory which neglects shear deformation and distortional effects,

(2) Abaqus beam model solution based on two-noded beam B31OS element with seven degrees of freedom per node (i.e., three translations, three rotations and warping deformation) which accounts for shear deformation only for bending but disregards shear deformation due to warping deformation and distortional effects,

(3) Abaqus shell model solution based on shell S4R element (four-noded doubly curved shell element with six degrees of freedom per node, i.e., three translations and three rotations) which incorporates the effects of shear deformation and distortion of the cross-section.

The material properties used in all examples are; $E = 200GPa$ and $G = 77GPa$.

Example (1): Coupled Bending-Torsional Static Solution

A 3.0m cantilever thin-walled beam with a channel asymmetric open section subjected to uniformly distribute transverse force $q_y(z) = 12.0\,kN/m$ applied along the beam axis is considered. The principal coordinates are inclined through an angle $\beta = -17.14^o$ (Figure 2). The centroidal coordinates in the global coordinate system are $(C_x, C_y) = (20mm, 60mm)$, while the coordinates of the shear centre S_c of the section along the principal axes (X, Y) are $(X_s, Y_s) = (-42.83mm, -10.29mm)$. The properties for the channel-section with respect to the principal coordinate system through the centroid C are provided in Table (1).This example is aimed at validating the accuracy of the present finite element formulations.

Table (1): Geometric and properties of asymmetric thin-walled J-section

$A = 0.20\times10^4 mm^2$	$I_{xx} = 3.723\times10^6 mm^4$	$I_{yy} = 0.878\times10^6 mm^4$
$J = 0.571\times10^5 mm^4$	$C_w = 0.861\times10^9 mm^6$	$D_{xx} = 11.65\times10^4 mm^2$
$D_{yy} = 8.348\times10^4 mm^2$	$D_{hx} = -49.15\times10^3 mm^3$	$D_{hy} = -1.794\times10^2 mm^3$
$D_{xy} = 1.127\times10^4 mm^2$	$D_{\omega\omega} = 3.222\times10^6 mm^4$	

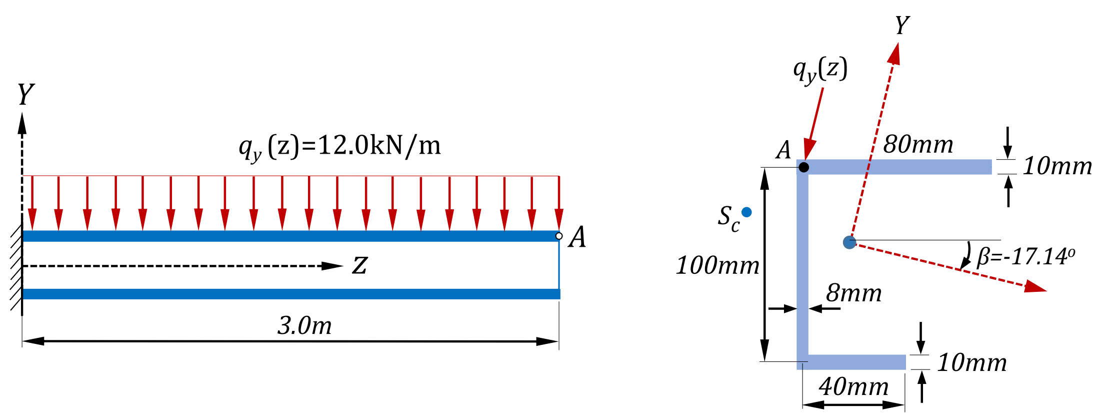


Figure 2: A cantilever beam with asymmetric C-section under distributed transverse force

Under the present finite element formulation, the nodal degrees of freedom are obtained using a single finite beam element with twelve-degrees of freedom per element. In Abaqus model solution, a total of 3,300 S4R shell elements (≈20,840 dof) are used (eight and four elements per upper and lower flanges, respectively, ten elements along web height and one hundred-fifty along the beam axis).

Coupled Flexural-Lateral-Torsional Static Response

The static response analysis of the cantilever beam of asymmetric cross-section under given distributed transverse force is provided in Table (2). The nodal displacement results obtained from the present finite element formulation based on using a single beam element (12 degrees of freedom) are found identical to the closed-form solution. It is seen that the present results for maximum nodal displacements and rotations are in excellent agreement with the corresponding results based on Abaqus shell element model using 3,300 shell elements. Abaqus results slightly differ from the present solution by 0.036%-5.73% and by 0.37%-8.75% from Vlasov solution. The first differences are due to distortional effects which are not included in the present finite element formulation, while the second differences are due to shear deformation and distorsional effects which are not captured in Vlasov model solution. Given the non-symmetry of the cross-section, the formulations suggest that, in general, all six degrees of freedom (i.e., displacement and rotation functions) are fully coupled. However, in the static results of the present cantilever example under transverse load, it is observed that the bending rotation angle θ_y about the Y axis vanish in all three solutions [24].

Table 2: Static results for coupled bending-torsional response for cantilever asymmetric C-section

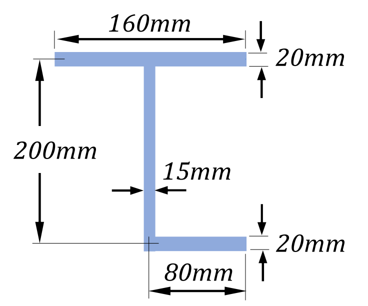
Variable	Abaqus S4R [1] (20,840dof)	Present Finite element [2] (12dof)	Vlasov Solution [3] (closed-form)	Present Difference =[1-2]/1	Vlasov Difference =[1-3]/1
u_A (mm)	2.633	2.784	2.845	-5.73%	-8.05%
v_A (mm)	-166.2	-165.6	-165.1	0.36%	0.66%
θ_x (10^{-3}rad)	72.98	72.71	72.52	0.37%	0.63%
θ_z (10^{-3}rad)	-134.7	-128.6	-127.9	4.53%	5.05%
ψ (10^{-3} rad/mm)	6.843	6.568	6.446	4.02%	8.75%

Example (2) Finite Element Formulation

A 10.0m clamped-clamped thin-walled beam having an asymmetric J-section subjected to uniformly distributed twisting moment $m_z(z)=5.0kNm/m$, and two concentrated transverse forces $P_{y1}(4m)=12.0kN$ and $P_{y2}(6m)=12.0kN$ is considered as shown in Figure (3). The centroidal coordinates are $C_x=8.205mm$ and $C_y=120.5mm$, while the coordinates of the shear centre S_c along the principal coordinates (X,Y) are: $(X_s,Y_s)=(-23.89mm,42.24mm)$, where the orientation of principal direction is $\beta=9.46^o$. The geometric and properties of *J*-section are provided in Table (3). It is required to assess the accuracy and efficiency of the present finite element formulation in analysing the coupled bending-torsional static response.

Table 3: Geometric and properties of asymmetric thin-walled J-section

$A = 0.78\times10^4\,mm^2$	$I_{xx} = 56.16\times10^6\,mm^4$
$I_{yy} = 8.489\times10^6\,mm^4$	$J = 0.865\times10^6\,mm^4$
$C_w = 57.0\times10^9\,mm^6$	$D_{xx} = 47.51\times10^4\,mm^2$
$D_{yy} = 30.49\times10^4\,mm^2$	$D_{xy} = -2.92\times10^4\,mm^2$
$D_{hx} = -3.71\times10^4\,mm^3$	$D_{hy} = 1.30\times10^4\,mm^3$
$D_{\omega\omega} = 47.14\times10^6\,mm^4$	



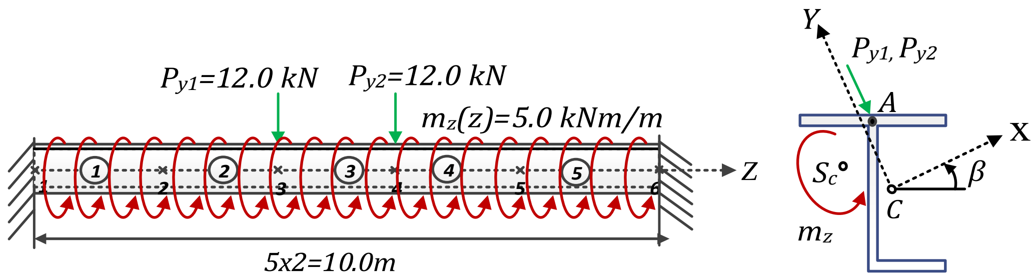


Figure 3: A clamped-clamped asymmetric thin-walled beam under static forces

FINITE ELEMENT FORMULATION

In order to demonstrate the ability of the finite element developed in this paper, the six nodal degrees of freedom per node associated with coupled flexural-lateral-torsional-warping static response are obtained using five beam elements (36 degrees of freedom). In this example, the results based on the present formulation are compared with Abaqus shell and beam element models. To eliminate the discretization errors and attain the required accuracy of the solution, three finite element solutions are provided for the problem. The first solution is based on the Abaqus shell element model where the beam is subdivided into 200 S4R elements along the longitudinal direction, 10 elements long height of the web, and eight and four elements along the width of the upper and lower flanges, respectively. The shell model consists of 4,400 S4R shell element with six degrees of freedom per node, which leads to approximately 27,740 degrees of freedom. The second solution is based on Abaqus finite beam model of one-hundred B31OS beam elements in which a total of 700 degrees of freedom were needed to attain the required accuracy. The third solution is based on the present finite element formulation. The beam is subdivided into only five beam elements along the beam span, i.e., the beam model have only 30 degrees of freedom.

The static analysis results for the nodal transverse displacement v_A of point A (Figure 3), related rotation θ_x, lateral displacement u_A, twist angle θ_z and warping deformation ψ plotted against the beam axis z are shown in Figure (4a-d). Again, similar observation is found, in which the bending rotation θ_y nearly vanished in all three solutions [24]. The figures demonstrate excellent agreement between the nodal displacement functions predicted by the present finite element model (using 30 degrees of freedom) and the Abaqus finite element models, the shell model (using 27,740 degrees of freedom) and the beam model (using 700 degrees of freedom). The slightly deviation between the results are attributed to cross-section distortional effects which are captured in Abaqus shell element solution but not in the other two solutions. The computational effort in the present finite element solution is several orders of magnitude less than that of other solutions. This is a natural outcome of the fact that the present finite element solution is based on the shape functions, which exactly satisfy the homogenous form of the coupled static equations, which eliminates any mesh discretization errors encountered in finite element formulations.

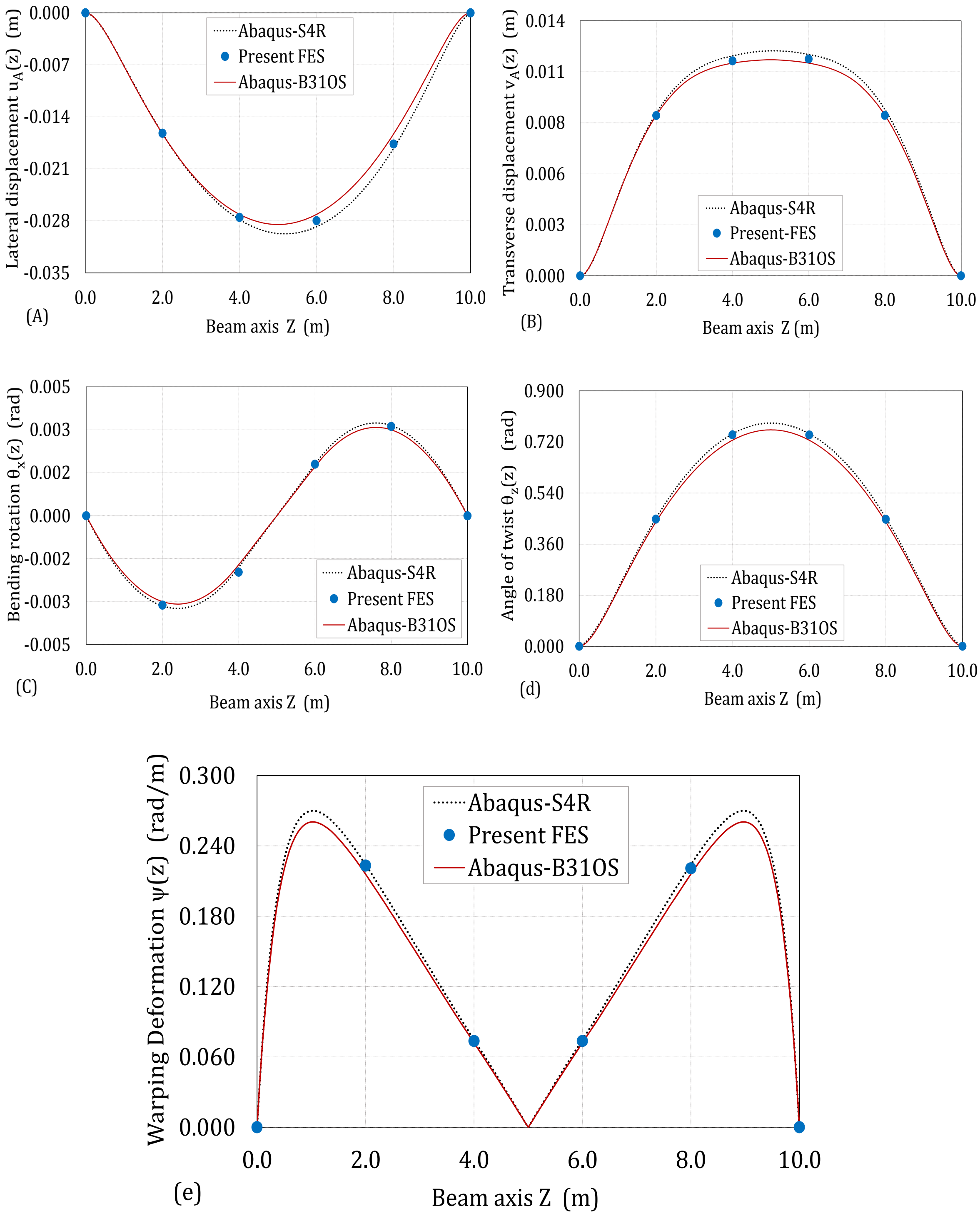


Figure 4: Static analysis for clamped-clamped asymmetric beam under various static forces

SUMMARY AND CONCLUSIONS

1. The static equations and related boundary conditions are derived for thin-walled members with asymmetric open cross-sections under general forces and moments. The formulation captures the shear deformation effects caused by bending and warping, and bending-torsional coupling effects caused by nonsymmetry of the cross-section.
2. The new two-noded finite beam element developed for thin-walled members having asymmetric open cross-sections is based on exact shape functions which exactly satisfied the exact homogeneous solution of the coupled static equations.
3. A superconvergent finite beam element is then formulated based on the exact shape functions.
4. The new beam element exhibits no discretization errors and generally provides excellent results with Abaqus beam and shell model solutions while keeping the number of degrees of freedom a minimum.
5. Comparison with established finite element and analytical solutions shows the validity, accuracy and efficiency of the present finite element formulation.
6. Comparisons with the Vlasov beam theory shows the importance of the shear deformation effects on the coupled static response.

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