GALERKIN FINITE ELEMENT ANALYSIS FOR BENDING OF ANISOTROPIC PLATES

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الملخص

تهتم هذه الورقة بتقديم طريقة جاليركن المبنية على المتبقيات الموازنة لاستخدامها كنهج للتحليل باستخدام العناصر المتناهية. لقد تم اختبار دقة هذا النهج بالمقارنة بما توفر من حلول رياضية دقيقة وكذلك بالمقارنة بنتائج بعض برامج الحاسوب ذات العلاقة والمعتمدة دوليا. لقد تم تطبيق هذا النهج لدراسة تصرف الصفائح المعدنية عند تعرضها لأحمال حني. في هذه الورقة تم أيضا استنتاج الصيغ الرياضية الخاصة بدوال الشكل والتجريب لعدد من العناصر البارامترية المتناهية. لقد تمت الحسابات باستخدام برنامج حاسوب تمت كتابته لهذا الغرض.

ABSTRACT

This paper is concerned with the introduction of the Galerkin Weighted Residual Method as an approach to the finite element analysis. The soundness of such approach, for the solution of engineering problems, is checked using exact solution and universally approved finite element packages. This approach is implemented for the study of anisotropic plates when subjected to bending. The shape as well as the trial functions were developed and used for a number of high-order parametric finite elements. For the analysis, an office made computer program package was developed and used.

INTRODUCTION

Structural plates have a multitude of applications in the aerospace and construction industries; hence, many investigators gave attention to the analysis of plates in flexure in order to get approximate or exact solutions to rather simple and homogeneous plates. A general solution scheme applicable to arbitrary shape and loading of a single as well as multicomponent plate structures, is the most desired. Finite element methods will provide such general solution scheme.

Despite the fact that the Galerkin approach to finite element is very powerful, easy to understand, and effectively applicable to the spectrum of engineering problems, no much attention was given to it in the literature. This paper is devoted for the presentation of such approach as well as the demonstration of its adequacy for the solution of bending of anisotropic plates. In this paper a four-node twenty-four DOF, (4N-24DOF), Sub-parametric Quadratic Element, is developed and used for the analysis of a single-layer specially orthotropic configuration of plates.

ANISOTROPIC PLATE BENDING THEORY

A homogeneous material displays identical properties throughout. If the properties are identical in all directions at a point, the material is termed isotropic. A non-isotropic or anisotropic solid such as wood displays direction-dependent properties, e.g, greater strength in a direction parallel to the grain than perpendicular to the grain. Single crystals also display pronounced anisotropy, manifesting different properties along the various crystallographic directions. For specially orthotropic plates, the principal material coordinates coincide with those of the plate.

Constitutive Relations [1]

Consider a plate prior to deformation, shown in Figure(1a), in which the x, y plane coincides with the midsurface and hence the z deflection is zero. However, due to external loading, the midsurface at any point x_A, y_A suffers deflection w, as shown in Figure(1b).

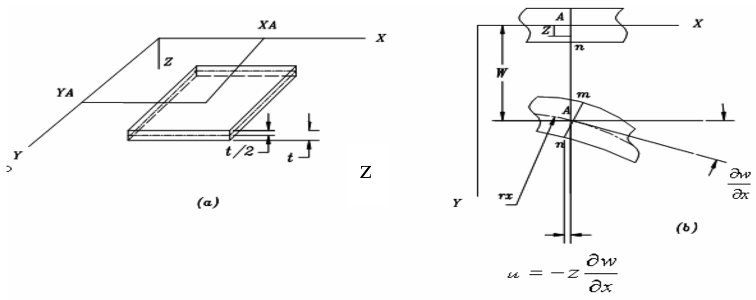

Figure 1: Geometry of undeformed and deformed (x, z) plane for classical plate theory

In plane stress condition, the strain form of the constitutive relation could be expressed in a matrix form as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad \text{or} \quad \{\varepsilon\} = [S]\{\sigma\} \tag{1}$$

Where; $[S]$, is the orthotropic compliance matrix.

On the other hand, the stress form of the constitutive relation is expressed as

$$\{\sigma\} = [Q]\{\varepsilon\}$$

$$\{Q\} = [S]^{-1} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \tag{2}$$

Where; $[Q]$, is the orthotropic stiffness matrix.
For plate bending, the strains could be written as:

$$\{\varepsilon\} = -z \left\{ \frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad -2\frac{\partial^2 w}{\partial x \partial y} \right\}^T \tag{3}$$

From Maxwell's reciprocal theorem

$$v_{21} E_1 = v_{12} E_2 \tag{4}$$

The Bending moments[1]

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{\frac{-t}{2}}^{\frac{t}{2}} \{\sigma\} z dz = \begin{bmatrix} E'_x & E'' & 0 \\ E'' & E'_y & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = -[D] \left\{ \frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad -2\frac{\partial^2 w}{\partial x \partial y} \right\}^T \tag{5}$$

where; $[D] = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix}$ is the anisotropic rigidity matrix.

The coefficients of matrix $[D]$ are defined as

$$D_x = \frac{E'_x t^3}{12}, D_y = \frac{E'_y t^3}{12}, D_1 = \frac{E'' t^3}{12}, D_{xy} = \frac{G t^3}{12} \tag{6}$$

Where; $E'_x = \frac{E_1}{1 - v_{12} v_{21}}, E'_y = \frac{E_2}{1 - v_{12} v_{21}}$, $E'' = \frac{v_{21} E_1}{1 - v_{12} v_{21}} = \frac{v_{12} E_2}{1 - v_{12} v_{21}}$, and $G = G_{12}$

The Equilibrium Equation [2]

Referring to Figure (2), the governing equation in the bending moment form:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + p = 0 \tag{7}$$

Where; the body forces are assumed to be negligible relative to the surface loading. or in a matrix form:

$$\left\{ \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} \quad -2\frac{\partial^2}{\partial x \partial y} \right\} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} + p = 0.$$, or

$$\left\{ \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} \quad -2\frac{\partial^2}{\partial x \partial y} \right\} \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \left\{ \frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad -2\frac{\partial^2 w}{\partial x \partial y} \right\}^T + p = 0 \tag{8}$$

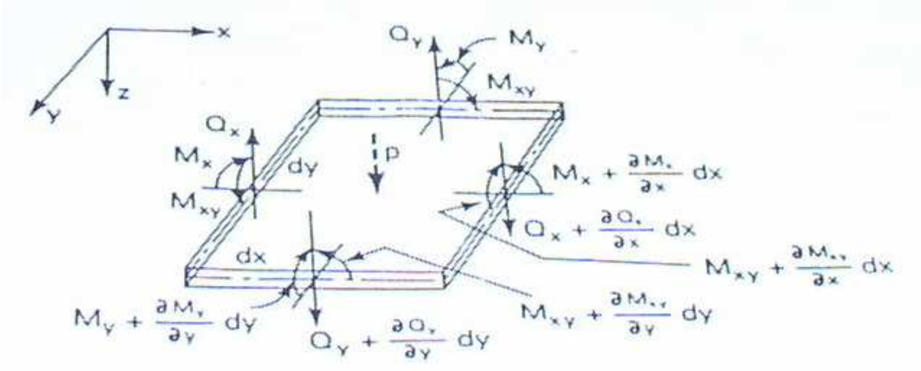

Figure 2: positive moments, shear forces, and lateral load on plate element

The Governing equation in deflection form, for orthotropic and elastic material [1]:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + p = 0$$

Where;

$$H = D_1 + 2D_{xy} \tag{9}$$

The shearing forces:

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = -\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} \end{Bmatrix} \begin{bmatrix} D_x & H \\ D_y & H \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \end{Bmatrix} \tag{10}$$

The governing equation in shear forces form:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \tag{11}$$

FINITE ELEMENT ANALYSIS

The finite element method (FEM) is a computer–aided mathematical technique for obtaining approximate numerical solution to the abstract equations of calculus that predicts the response of physical systems subjected to external influences. The finite element formulation involves construction of a trial solution, application of an optimization criterion, and estimation of accuracy. For optimizing the trial solution, there are two types of optimizing criteria that have played a dominant role in the FEM, namely, the methods of weighted residual (MWR), which are applicable when the governing equations are differential equations, and Ritz variational method (RVM), which is applicable, when the governing equations are variational (integral) equations.

The Galerkin Approach [3]

The Galerkin method is a form of MWR which is used as an optimizing criterion for the FEM formulation. The Galerkin approach does not need or use energy functional and can thus be applied to equations where RVM can not.

Considering the governing differential equation for a given physical system to be in the following form:

$$A(w) = f \quad \text{in the domain } \Omega \tag{12}$$

Where A is a linear or nonlinear differential operator, and the boundary conditions (**BCS**) are set in the form

$B_1(w) = \hat{w}$ on Γ_1 $\quad B_2(w) = \hat{g}$ on Γ_2

The residual of the governing equation is written as

$$R = A(w_a) - f \neq 0 \tag{13}$$

And the Galerkin weighted residual equation is expressed as

$$\int_\Omega N_i R\ (w_a, f) dV = 0, \quad \mathrm{i} = 1,2,.........,\mathrm{n} \tag{14}$$

Where; N_i are the trial functions, and w_a is the approximate solution of w ,

$$w_a = \sum_{j=1}^{n} N_j w_j \tag{15}$$

Finite element formulation for plate bending [4]

The governing equation in the form of shear forces, Eq. (11), would be used as the differential equation, thus, the residual equation is set as

$$R = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p \tag{16}$$

The Galerkin weighted residual equation would be

$$\int_A R N_i dA = 0 \tag{17}$$

The finite element equations would be in the form

$$[K]\{w\} = \{F\} \tag{18}$$

Where;

$$w = \sum_{j=1}^{n} N_j w_j \tag{19}$$

$$K_{ij} = \int_A -\left[D_x \frac{\partial^2 N_i}{\partial x^2} \frac{\partial^2 N_j}{\partial x^2} + D_1 \frac{\partial^2 N_i}{\partial x^2} \frac{\partial^2 N_j}{\partial y^2} + 4 D_{xy} \frac{\partial^2 N_i}{\partial x \partial y} \frac{\partial^2 N_j}{\partial x \partial y} + D_y \frac{\partial^2 N_i}{\partial y^2} \frac{\partial^2 N_j}{\partial y^2} + D_1 \frac{\partial^2 N_i}{\partial y^2} \frac{\partial^2 N_j}{\partial x^2} \right]$$

$$F_i = -\oint_S N_i Q_n ds + \oint_S M_n \frac{\partial N_i}{\partial n} ds - \oint_S N_i \frac{\partial M_{ns}}{\partial s} ds + \left[N_i M_{ns} \right]_s - \int_A N_i p dA \tag{20}$$

The above equations are developed with the aid of the following relations,

$$Q_n = Q_x n_x + Q_y n_y$$

$$M_n = M_x n_x^2 + M_y n_y^2 + 2 M_{xy} n_x n_y$$

$$\frac{\partial}{\partial x} = n_x \frac{\partial}{\partial n} - n_y \frac{\partial}{\partial s} \tag{21}$$

$$\frac{\partial}{\partial y} = n_x \frac{\partial}{\partial s} - n_y \frac{\partial}{\partial n}$$

Where; n and s are the coordinates normal and tangent to the boundaries, while n_x , and n_y are direction cosines.

Parametric Elements

The establishment of shape and/or trial functions is greatly facilitated using localized, natural, coordinate system. In this system a master, or parent element, shown in Figure (3), is adopted. This parent element has four nodes at its four corners, with the origin at its centre, while the sides expand to $\xi = \pm 1,\ \eta = \pm 1$. The successful mapping, transformation, entails that the relation between the global and the natural coordinate systems must be one-to-one.

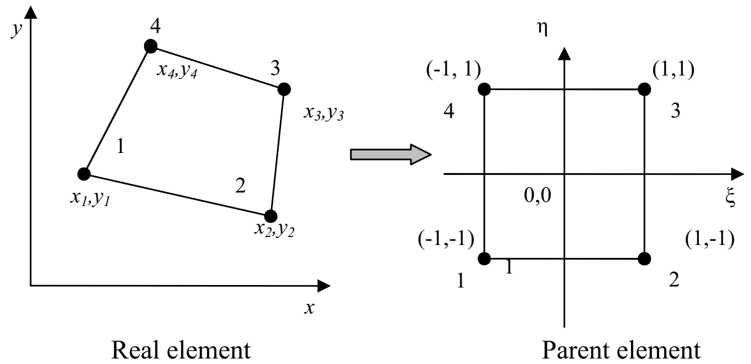

Figure 3: The real and parent elements

Construction of shape and trial functions [4]

The construction of shape and trial functions is performed pursuing the following steps:

- Choosing the polynomial basis $\langle P(\xi,\eta)\rangle$
- Evaluating the nodal matrix $[P_n] = \lfloor P_j(\xi_i,\eta_i)\rfloor_{i,j=1,2,3,.....,n_d}$ (22)
- Inverting $[P_n]$
- Computing f $\langle N\rangle$ where $\langle N\rangle = \langle P(\xi,\eta)\rangle [P_n]^{-1}$ (23)

It should be noted that the chosen polynomials ought to be as complete as possible. In a two dimensional continuum, the field variable, $w(x, y)$, could be written in a complete polynomial as follows:

$$w(x,y;a) = \sum_{k=1}^{T} a_k x^i y^j,\quad i + j \le m,\quad i,j = 0,\ 1,\ 2,.............,m \tag{24}$$

Where; $T = \frac{(m+1)(m+2)}{2}$, is the number of terms in the polynomial, and m is the degree of the polynomial.

Shape Function for 2-D Quadratic Serendipity lement

Considering the quadratic element, shown in Figure (3), the shape function could be constructed pursuing the following steps:

a) ***Choice of polynomial basis*** $[\overline{P}]$

For the quadratic element, $m = 2$, thus T = 6, while only four terms are required for the polynomial for such element, which means that a complete polynomial can not be used. The best choice is a bilinear polynomial since it would respect both, symmetry and inter-element continuity.

$$\langle \overline{P} \rangle = \langle 1, \quad \xi, \quad \eta, \quad \xi\eta \rangle \tag{25}$$

b) Evaluation of $[\overline{P}_n]$

c)

$$[\overline{P}_n] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{matrix} \longrightarrow node_1\ (\xi,\eta) = (-1,-1) \\ \longrightarrow node_2\ (\xi,\eta) = (+1,-1) \\ \longrightarrow node_3\ (\xi,\eta) = (+1,+1) \\ \longrightarrow node_4\ (\xi,\eta) = (-1,+1) \end{matrix} \tag{26}$$

d) *Inversion of* $[\overline{P}_n]$

$$[\overline{P}_n]^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

e)Expression of $\langle \overline{N} \rangle$

$$\langle \overline{N} \rangle = \langle \overline{N}_1 \quad \overline{N}_2 \quad \overline{N}_3 \quad \overline{N}_4 \rangle = \langle \overline{P} \rangle [\overline{P}_n]^{-1} \tag{27}$$

$$\langle \overline{N} \rangle = \frac{1}{4} \langle (1-\xi)(1-\eta); \quad (1-\xi)(1-\eta); \quad (1-\xi)(1-\eta); \quad (1-\xi)(1-\eta) \rangle \tag{28}$$

The Jacobean $[J]$ can be obtained by applying the chain role, thus

$$\begin{Bmatrix} \dfrac{\partial \overline{N}}{\partial \xi} \\ \dfrac{\partial \overline{N}}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial \overline{N}}{\partial x} \\ \dfrac{\partial \overline{N}}{\partial y} \end{Bmatrix} \text{, where; } [J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \text{, and } [J]^{-1} = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \tag{29}$$

Where;

$$x = \sum_{i=1}^{n} \overline{N}_i(\xi,\eta) x_i, \qquad y = \sum_{i=1}^{n} \overline{N}_i(\xi,\eta) y_i,$$

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial \overline{N}_i}{\partial \xi} x_i, \ \frac{\partial x}{\partial \eta} = \sum_{i=1}^{n} \frac{\partial \overline{N}_i}{\partial \eta} x_i, \ \frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial \overline{N}_i}{\partial \xi} y_i, \text{ and } \frac{\partial y}{\partial \eta} = \sum_{i=1}^{n} \frac{\partial \overline{N}_i}{\partial \eta} y_i \tag{30}$$

Trial function for 2-D quadratic serendipity element (4N-24DOF)

A study of mesh refinement effect on accuracy for different types of elements reveals that solution accuracy depends on the number and type of element involved. As could be inferred from Figure (4), the accuracy improves by increasing the number of elements up to a certain value beyond which accuracy starts to decline.

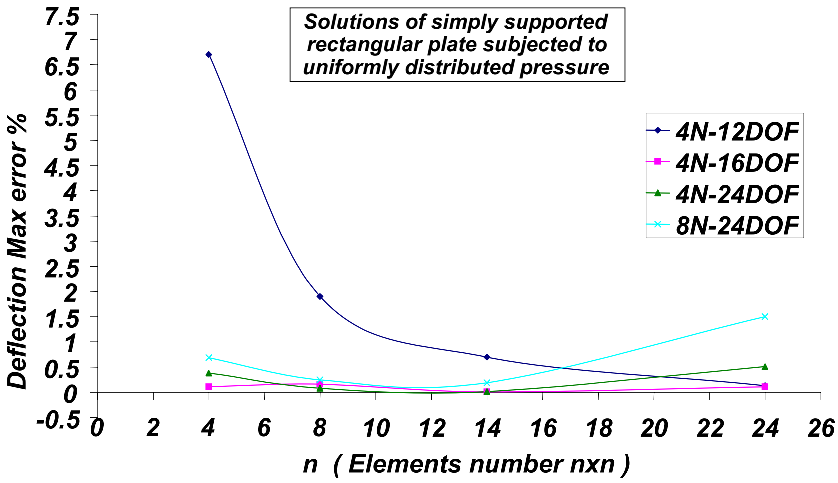

Figure 4: effect of mesh refinement on accuracy [5]

Elements type 4N-16DOF and 4N-24DOF prove to be the best [5]. For this reason, this section is devoted for the construction of the trial function for the latter element. This type of element, shown in Figure (5), is one of the conformal plate bending elements. The element has twenty four degrees of freedom, six degrees of freedom per node [6], namely, w, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$ and $\frac{\partial^2 w}{\partial x \partial y}$.

Once again, the trial function for this type of element could be constructed following the same steps as for the construction of shape function.

- The polynomial basis is expressed as

$$\langle P \rangle = \langle 1 \quad \xi \quad \eta \quad \xi^2 \quad \xi\eta \quad \eta^2 \quad \xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3 \quad \xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4 \quad \xi^5 \quad \xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4 \quad \eta^5 \quad \xi^5\eta \quad \xi\eta^5 \quad \xi^3\eta^3 \rangle \tag{31}$$

The polynomial basis and its derivatives will constitute the global polynomial matrix, P_G, thus,

$$\bullet\; u_i = \begin{Bmatrix} w_i \\ \partial w_{i,x} \\ \partial w_{i,y} \\ \partial^2 w_{i,x} \\ \partial^2 w_{i,y} \\ \partial^2 w_{i,xy} \end{Bmatrix}; i = 1,2,3,4 \qquad \bullet\; \{u_i\}_{\xi,\eta} = \begin{Bmatrix} w_i \\ \partial w_{i,\xi} \\ \partial w_{i,\eta} \\ \partial^2 w_{i,\xi} \\ \partial^2 w_{i,\eta} \\ \partial w_{i,\xi\eta} \end{Bmatrix}; i = 1,2,3,4$$

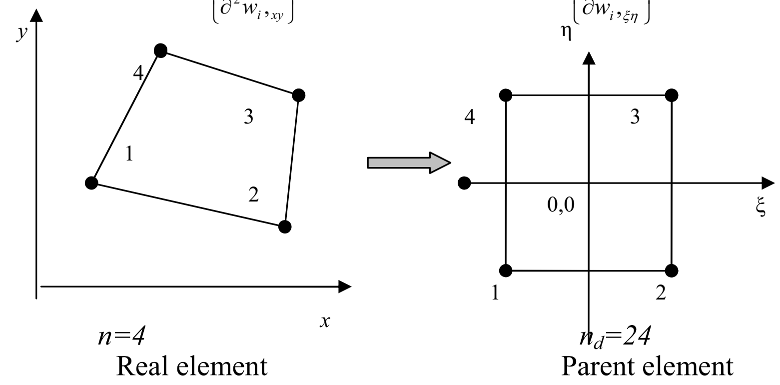

Figure 5: 4N-24DOF Element

$$\langle P_G \rangle = \langle P \quad P_\xi \quad P_\eta \quad P_{\xi^2} \quad P_{\eta^2} \quad P_{\xi\eta} \rangle^T \tag{32}$$

- The nodal matrix is written as

$$[P_n] = [P_G(\xi_i, \eta_i)]_j \quad i, j = 1.,2,3,\ldots.,n_d \tag{33}$$

- Computation of Global $\langle N_G \rangle$

The interpolation function N and its derivatives constitutes the elements of the global matrix, N_G, so that,

$$[N_G] = \langle \langle N \rangle \quad \langle N_\xi \rangle \quad \langle N_\eta \rangle \quad \langle N_{\xi^2} \rangle \quad \langle N_{\eta^2} \rangle \quad \langle N_{\xi\eta} \rangle \rangle^T = [P_G][P_n]^{-1} \tag{34}$$

The nodal matrix and its inverse are presented in the Appendix

For rectangle elements with straight sides, it is generally possible to obtain exact results by numerical integration using Gauss sampling points. The stiffness and load matrix coefficients, for the element, would be written as [5],

$$K_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n} \left(-|J| w_k w_l \left(D_x \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x^2} + D_1 \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2} + 4D_{xy} \frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial x \partial y} + D_y \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2} + D_1 \frac{\partial^2 N_i}{\partial y^2} \right.\right. \tag{35}$$

Where; w_k, w_l are Gauss weights, and n is the number of sampling points.

$$F_i = \sum_{k=1}^{n} \sum_{l=1}^{n} \left(-|J| w_k w_l (pN_i)\right)_{\xi_l,\eta_k} - \oint_S N_i Q_n ds + \oint_S M_n \frac{\partial N_i}{\partial n} ds - \oint_S N_i \frac{\partial M_{ns}}{\partial s} ds + \left[N_i M_{ns}\right]_s \tag{36}$$

The second derivatives are expressed as:

$$\begin{Bmatrix} \dfrac{\partial^2 N}{\partial \xi^2} \\ \dfrac{\partial^2 N}{\partial \eta^2} \\ \dfrac{\partial^2 N}{\partial \xi \partial \eta} \end{Bmatrix} = \begin{bmatrix} \left(\dfrac{\partial x}{\partial \xi}\right)^2 & \left(\dfrac{\partial y}{\partial \xi}\right)^2 & 2\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \xi} \\ \left(\dfrac{\partial x}{\partial \eta}\right)^2 & \left(\dfrac{\partial y}{\partial \eta}\right)^2 & 2\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \eta} \\ \dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta} & \dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 N}{\partial x^2} \\ \dfrac{\partial^2 N}{\partial y^2} \\ \dfrac{\partial^2 N}{\partial x \partial y} \end{Bmatrix} + \begin{bmatrix} \dfrac{\partial^2 x}{\partial \xi^2} & \dfrac{\partial^2 y}{\partial \xi^2} \\ \dfrac{\partial^2 x}{\partial \eta^2} & \dfrac{\partial^2 y}{\partial \eta^2} \\ \dfrac{\partial^2 x}{\partial \xi \partial \eta} & \dfrac{\partial^2 y}{\partial \xi \partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \\ \dfrac{\partial N}{\partial y} \end{Bmatrix} \tag{37}$$

CASE STUDY

The case that would be studied in this paper is the simply supported anisotropic square plate subjected to a uniformly distributed pressure. A computer code is developed, using the aforementioned Galerkin finite element equations, and used for the analysis. The results of such code are validated using exact solution for such case.

Input Data

The plate considered has the following dimensions:

Length, a = 1000 mm, width, b= 1000 mm, and the thickness, t = 20mm

The plate is subjected to a uniformly distributed pressure P = 0.1 N/mm^2

The material considered is Glass-Epoxy lamina that has the following properties [1]:

E_x = 53781 N/mm^2, E_y = 17927 N/mm^2, G_{xy} = 8965.5 N/mm^2, ν_{xy} = 0.25

Exact Solution

The exact solution is expressed in double trigonometric series form as [2]:

$$w = \frac{16 p_0}{\pi^6} \sum_m^{\infty} \sum_n^{\infty} \frac{\sin(m\pi x/a)\sin(n\pi y/b)}{mn[D_x (m/a)^4 + H \dfrac{2m^2 n^2}{a^2 b^2} + D_y (n/b)^4]} \tag{38}$$

Where; m, n=1,3,5,………

Results and Discussion

The results of the developed computer code, using the Galerkin approach of finite element, are compared with those of the exact solution, Eq (38). The code results excellently match the exact solution and demonstrated a maximum error of about 0.0078% over the solution domain.

Table 1: Results of Comparison

	E_X N/mm^2	E_Y N/mm^2	G_{XY} N/mm^2	w_{max} mm	$\varepsilon_{x\ max}$	$\theta_{x\ max}$ rad	$\sigma_{v,\ max}$ N/mm^2
Glass-Epoxy	53781	17927	8965.5	20.5782	0.004105	0.07040	100.1637
Glass	53781	53781	8965.5	13.6467	0.002650	0.04490	89.1488
Epoxy	17927	17927	8965.5	29.0910	0.005800	0.09690	104.007

The distribution of the deflection and Von Mesis equivalent stress, σ_v, using sixty four elements, are depicted in Figures (6) and (7) respectively.

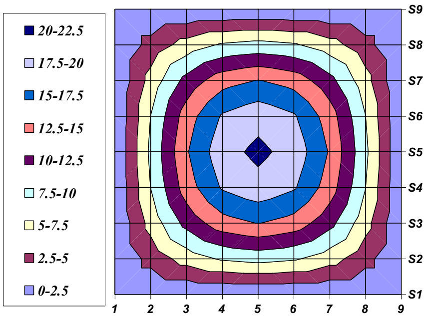

Figure 6: Distributions of the w (mm (N/mm^2)

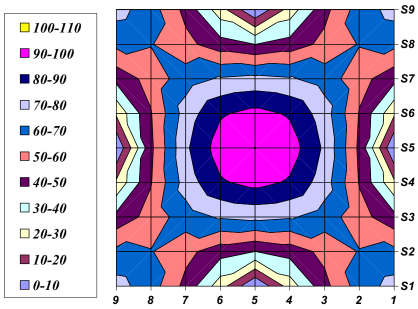

Figure 7: Distributions of the σ_v

The code is also capable of presenting the solution for isotropic plates. A comparison of the solutions of isotropic and anisotropic plates is made to study the effect of orientation dependency of material property on rigidity and stresses.

The behaviour of Epoxy-Glass as an orthotropic material is compared with that of an isotropic material having same mechanical properties as for glass for one case and as for epoxy for the other case. The results of such comparison, for a plate with the same dimension as indicated above, are depicted in Table (1).

It could be inferred from Table (1) that the rigidity parameters such as deflection and rotation are greatly affected by anisotropy, however anisotropy has little effect of equivalent stress.

CONCLUSION

The sole objective of this paper is the demonstration of the soundness of the Galerkin approach in solving complicated engineering problems. To that extent, the authors elected a special form of anisotropic plates, namely, specially orthotropic, and wrote the required computer code. The results demonstrated the effectiveness of such approach.

The element type (4N-24DOF) possesses high performance because it produces the best results for the field variables and for the stresses and strains results. This is equally true for other cases with plates of different shapes and loadings [5]. This is simply because the second derivatives are the field variables for this element type. The results also have indicated that the rigidity parameters are very sensitive to anisotropy while stresses are less sensitive.

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APPENDIX

This appendix is devoted for the presentation of the nodal matrix and its inverse for element type (4N-24DOF).

$$[P_n]=\left[\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrr}
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
0 & 1 & 0 & -2 & -1 & 0 & 3 & 2 & 1 & 0 & -4 & -3 & 2 & -1 & 0 & 5 & 4 & 3 & 2 & 1 & 0 & -5 & -1 & -3 \\
0 & 0 & 1 & 0 & -1 & -2 & 0 & 1 & 2 & 3 & 0 & -1 & -2 & -3 & -4 & 0 & 1 & 2 & 3 & 4 & 5 & -1 & -5 & -3 \\
0 & 0 & 0 & 2 & 0 & 0 & -6 & -2 & 0 & 0 & 12 & 6 & 2 & 0 & 0 & -20 & -12 & -6 & -2 & 0 & 0 & 20 & 2 & 6 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & -6 & 0 & 0 & 2 & 6 & 12 & 0 & 0 & -2 & -6 & -12 & -20 & 0 & 20 & 6 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & -2 & 0 & 0 & 3 & 4 & 3 & 0 & 0 & -4 & -6 & -6 & -4 & 0 & 5 & 5 & 9 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\
0 & 1 & 0 & 2 & -1 & 0 & 3 & -2 & 1 & 0 & 4 & -3 & 2 & -1 & 0 & 5 & -4 & 3 & -2 & 1 & 0 & -5 & -1 & -3 \\
0 & 0 & 1 & 0 & 1 & -2 & 0 & 1 & -2 & 3 & 0 & 1 & -2 & 3 & -4 & 0 & 1 & -2 & 3 & -4 & 5 & 1 & 5 & 3 \\
0 & 0 & 0 & 2 & 0 & 0 & 6 & -2 & 0 & 0 & 12 & -6 & 2 & 0 & 0 & 20 & -12 & 6 & -2 & 0 & 0 & -20 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & -6 & 0 & 0 & 2 & -6 & 12 & 0 & 0 & 2 & -6 & 12 & -20 & 0 & -20 & -6 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -2 & 0 & 0 & 3 & -4 & 3 & 0 & 0 & 4 & -6 & 6 & -4 & 0 & 5 & 5 & 9 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 5 & 4 & 3 & 2 & 1 & 0 & 5 & 1 & 3 \\
0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 5 & 1 & 5 & 3 \\
0 & 0 & 0 & 2 & 0 & 0 & 6 & 2 & 0 & 0 & 12 & 6 & 2 & 0 & 0 & 20 & 12 & 6 & 2 & 0 & 0 & 20 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 6 & 0 & 0 & 2 & 6 & 12 & 0 & 0 & 2 & 6 & 12 & 20 & 0 & 20 & 6 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 3 & 4 & 3 & 0 & 0 & 4 & 6 & 6 & 4 & 0 & 5 & 5 & 9 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
0 & 1 & 0 & -2 & 1 & 0 & 3 & -2 & 1 & 0 & -4 & 3 & -2 & 1 & 0 & 5 & -4 & 3 & -2 & 1 & 0 & 5 & 1 & 3 \\
0 & 0 & 1 & 0 & -1 & 2 & 0 & 1 & -2 & 3 & 0 & -1 & 2 & -3 & 4 & 0 & 1 & -2 & 3 & -4 & 5 & -1 & -5 & -3 \\
0 & 0 & 0 & 2 & 0 & 0 & -6 & 2 & 0 & 0 & 12 & -6 & 2 & 0 & 0 & -20 & 12 & -6 & 2 & 0 & 0 & -20 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 6 & 0 & 0 & 2 & -6 & 12 & 0 & 0 & -2 & 6 & -12 & 20 & 0 & -20 & -6 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 2 & 0 & 0 & 3 & -4 & 3 & 0 & 0 & -4 & 6 & -6 & 4 & 0 & 5 & 5 & 9
\end{array}\right]$$

$$[P_n]^{-1} = \left(\frac{1}{32}\right)\begin{bmatrix}
8 & 5 & 5 & 1 & 1 & 2 & 8 & -5 & 5 & 1 & 1 & -2 & 8 & -5 & -5 & 1 & 1 & 2 & 8 & 5 & -5 & 1 & 1 & -2 \\
-15 & -7 & -7 & -1 & -1 & -2 & 15 & -7 & 7 & 1 & 1 & -2 & 15 & -7 & -7 & 1 & 1 & 2 & -15 & -7 & 7 & -1 & -1 & 2 \\
-15 & -7 & -7 & -1 & -1 & -2 & -15 & 7 & -7 & -1 & -1 & 2 & 15 & -7 & -7 & 1 & 1 & 2 & 15 & 7 & -7 & 1 & 1 & -2 \\
0 & -6 & 0 & -2 & 0 & -2 & 0 & 6 & 0 & -2 & 0 & 2 & 0 & 6 & 0 & -2 & 0 & -2 & 0 & -6 & 0 & -2 & 0 & 2 \\
24 & 9 & 9 & 1 & 1 & 2 & -24 & 9 & -9 & -1 & -1 & 2 & 24 & -9 & -9 & 1 & 1 & 2 & -24 & -9 & 9 & -1 & -1 & 2 \\
0 & 0 & -6 & 0 & -2 & -2 & 0 & 0 & -6 & 0 & -2 & 2 & 0 & 0 & 6 & 0 & -2 & -2 & 0 & 0 & 6 & 0 & -2 & 2 \\
10 & 10 & 2 & 2 & 0 & 2 & -10 & 10 & -2 & -2 & 0 & 2 & -10 & 10 & 2 & -2 & 0 & -2 & 10 & 10 & -2 & 2 & 0 & -2 \\
0 & 8 & 0 & 2 & 0 & 2 & 0 & -8 & 0 & 2 & 0 & 2 & 0 & 8 & 0 & -2 & 0 & -2 & 0 & -8 & 0 & -2 & 0 & 2 \\
0 & 0 & 8 & 0 & 2 & 2 & 0 & 0 & -8 & 0 & -2 & 2 & 0 & 0 & 8 & 0 & -2 & -2 & 0 & 0 & -8 & 0 & 2 & -2 \\
10 & 2 & 10 & 0 & 2 & 2 & 10 & -2 & 10 & 0 & 2 & -2 & -10 & 2 & 10 & 0 & -2 & -2 & -10 & 2 & 10 & 0 & -2 & 2 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
-12 & -12 & -2 & -2 & 0 & -2 & 12 & -12 & 2 & 2 & 0 & -2 & -12 & 12 & 2 & -2 & 0 & -2 & 12 & 12 & -2 & 2 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -2 \\
-12 & -2 & -12 & 0 & -2 & -2 & 12 & -2 & 12 & 0 & 2 & -2 & -12 & 2 & 14 & 0 & -2 & -2 & 12 & 2 & -12 & 0 & 2 & -2 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-3 & -3 & 0 & -1 & 0 & 0 & 3 & -3 & 0 & 1 & 0 & 0 & 3 & -3 & 0 & 1 & 0 & 0 & -3 & -3 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & -2 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & -2 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
-3 & 0 & -3 & 0 & -1 & 0 & -3 & 0 & -3 & 0 & -1 & 0 & 3 & 0 & -3 & 0 & 1 & 0 & 3 & 0 & -3 & 0 & 1 & 0 \\
3 & 3 & 0 & 1 & 0 & 0 & -3 & 3 & 0 & -1 & 0 & 0 & 3 & -3 & 0 & 1 & 0 & 0 & -3 & -3 & 0 & -1 & 0 & 0 \\
3 & 0 & 3 & 0 & 1 & 0 & -3 & 0 & -3 & 0 & -1 & 0 & 3 & 0 & -3 & 0 & 1 & 0 & -3 & 0 & 3 & 0 & -1 & 0 \\
2 & 2 & 2 & 0 & 0 & 2 & -2 & 2 & -2 & 0 & 0 & 2 & 2 & -2 & -2 & 0 & 0 & 2 & -2 & -2 & 2 & 0 & 0 & 2
\end{bmatrix}$$

NOMENCLATURE

The following symbols have a unique meaning as defined throughout this paper.

D	Flexural rigidity of the plate	R	Residual
M	Bending moment per unit length	T	Plate thickness
Q	Shear force per unit length	V	Poisson's ratio
P	Intensity of distributed load	E	Modulus of elasticity
W	Plate transverse deflection	G	Shear modulus
N	Trial function	1,2	Material coordinates
$\overline{N}$	Shape function	P_ξ	First derivative with respect to ξ
X, y	Real coordinates	P_ξ^2	Second derivative with respect to ξ and so on
η, ξ	Natural coordinates	Γ_1, Γ_2	Line portions of the boundary
T	Transpose	[K]	Stiffness matrix
n_x, n_y	Direction cosines	{F}	Load matrix
σ_v	Von-Mesis equivalent stress	$\theta_x = \frac{\partial w}{\partial y}$	Rotation about x-axis
ε_x	Normal strain in the x-direction		