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Contents

Ground State Properties of Frustrated Ferromagnetic Ising Chains in a Uniform-Random-External Magnetic Field.

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Abstract

The methods of Williams, Doman and Williams, and Doman and Habeeb are used to describe the ground state properties of infinite ferromagnetic Ising chains in a random field. Analytic results for the ground state energy, the magnetisation per spin in the direction of the local field and the entropy as a function of magnetic field are obtained. When the probability factor of the zero local field is zero, our results are in agreement with that obtained previously by Williams.

1. Introduction

In a recently publised work, Grinstein and Mukamel (1983) found analytic results for the free energy, magnetic structure factor, and Edwards-Anderson order parameter of a one-dimensional Ising model in a random magnetic field. They argue the importance of considering such a system in one dimension for helping to clarify the unresolved issues associated with this problem (random field prblem) in other dimensionalities ($d_c = 2$ and $d_c = 3$, see for instance Villain (1982), Niemi (1982)). At $T=0$ they considered that each random field which exceeds 2J acting on a spin to be effectively of infinite strength and hence it constrains the spin to point in the direction of that field ($J > 0$ is the ferromagnetic bond strength). This model is experimentally applicable to the helix-coil transitions in DNA (deoxyribonucleic acid) and to quasi-one dimensional magnets, such as CsCoCl3, Azbel and Rubinstein (1983). Grinstein and Mukamel, Azbel and Rubinstein have examined the correlation function for the system.

In the following work we examine an infinite ferromagnetic Ising chain in a uniform randomly distributed external magnetic field at zero temperature. To avoid what seems to be unphysical assumption of the random field we define its distribution in a different way. The whole system is divided into finite subchains separated by terminators. A terminator could be defined as the simplest group of spins whose direction is independent of that of its neighbours, but depends on the value of the magnetic field. The behaviour of the system is now dominated by these subchains and so we take into account all possible configurations for such subchains. The idea of superspins and superbonds introduced by Williams (1981) and applied later to examine the properties of many Ising systems in one dimension by Doman and Williams (1982), Doman (1982) and Doman and Habeeb (1983) is used. The ground state energy, the magnetisation per spin in the direction of the local field and the entropy as a function of magnetic field for the system are evaluated. The magnetisation per spin in the direction of the local field and the entropy behaviour are shown graphically.

2. The Hamiltonian.

The considered system has the Hamiltonian,

$$H = -J \sum_i \sigma_i \sigma_{i+1} - B \sum_i \sigma_i \tau_i , \qquad (1)$$

where σ_i is spin variable ($\sigma_i = \pm 1$), J is the ferromagnetic bond strength ($J > 0$), B is the uniform random external magnetic field ($B > 0$) and the parameter τ_i is distributed according to the prbability equation:

$$\underline{p}(\tau_i = 1) = x$$

$$\underline{p}(\tau_i = 0) = y$$

$$\underline{p}(\tau_i = -1) = 1 - x - y \qquad (2)$$

The spins with zero random external magnetic field are defined as those with $\tau_i = 0$. Also, any two successive spins with $\tau_i \neq 0$ may be separted by a set of spins with $\tau_i = 0$.

3. The Model in the Range of Magnetic Field $B > 2J$ and $J < B < 2J$.

In the range of the field $B > 2J$ the terminators which separate the subchains are defined as single spins with $\tau_i \neq 0$ pointing in the same direction of its own field (i.e. $\sigma_i \tau_i = 1$ for such spins) and may be separated by spins with $\tau_i = 0$. Hence there will be frustrated ferromagnetic bond between any two opposite spins with $\tau_i \neq 0$. The frustrated bond has a degeneracy equal to the number of spins with $\tau_i = 0$ in between the terminators plus one. All the bonds are satisfied when the terminators are parallel.

Let N be the total number of spins in the system. Then the total ground state bond energy per spin is given by

$$E_{bond} = -J + 2J \times \text{fraction of the frustrated bond.} \qquad (3)$$

A pair of spins with $\tau_i \neq 0$ pointing in opposite direction has a probability factor of $2 \cdot \frac{x(1-x-y)}{1-y}$ found by including the possibility that they are *separated by $\tau_i = 0$ spins. Therefore,*

$$E_{bond} = -J + 4J \frac{x(1-x-y)}{1-y} \quad (4)$$

The total filed energy per spin, E_F, is given by :

$$E_F = -(1-y)B \quad (5)$$

Thus the total ground state energy per spin, E_o, is:

$$E_o = -J + \frac{4J\, x(1-x-y) - (1-y)^2 B}{(1-y)}, \quad B > 2J. \quad (6)$$

There will be a degeneracy in the position of the frustrated bond between any two opposite pointing spins with $\tau_i \neq 0$ separated by at least one $\tau_i = 0$ spin. For $m(\tau_i = 0)$ spins positions between two oppositely pointing $\tau_i \neq 0$ spins the degeneracy is $(m+1)$. The corresponding probability factor is $2x(1-x-y).y^m$ and the contribution to the entropy is $2x(1-x-y)y^m.\log(m+1)$. Hence the total entropy per spin So is

$$S_o = 2x(1-x-y) \sum_{m=1}^{\infty} y^m \log(m+1), \qquad B > 2J$$

In the range $J < B < 2J$ the terminators are defined as a pair of parallel ($\tau_i \neq 0$) spins which may be separated by ($\tau_i = 0$) spins and both are pointing in the direction of its own field. Hence each type of subchain has two possible configurations depending upon the direction of the terminators.

As B reduces to this new range, $J < B < 2J$ of the field, some isolated ($\tau_i \neq 0$) spins will change their direction and point against their own field. All frustrated bonds are relieved between terminators pointing in the same direction. So, in a section of $2n+1$ ($\tau_i \neq 0$) spins at most $n+1$ ($\tau_i \neq 0$) spins can be flipped against their own fields. For terminators pointing in opposite directions there will remain one frustrated bond between them. So, in a section of $2n(\tau_i \neq 0)$ spins at most $n(\tau_i \neq 0)$ spins can be flipped against their own fields. Therefore, the ground state has a degeneracy of

$$(m_1+1) + (m_3+1) + (m_5+1) + \ldots + (m_{2n+1}+1),$$

due to the different possible positions of the frustrated bond, where m_i is the number of ($\tau_i = 0$) spins that separated the $(i-1)^{th}$ and i^{th} ($\tau_i \neq 0$) spins.

Define $P(n, \{m\})$ as the probability factor corresponding to a subchain composed of $n(\tau_i = 0)$ spins. The probability factor for a terminator being up is $\frac{x^2}{1-y}$ and for beingand for being down is $\frac{(1-x-y)^2}{1-y}$. Hence we find,

$$\underline{P}(2n, \{m\}) = 2\left[\frac{x(1-x-y)}{1-y}\right]^2 [x(1-x-y)]^n \prod_{i=1}^{2n+1} y^{m_i} \quad (8)$$

$$P(2n+1, \{m\}) = \frac{x^3 + (1-x-y)^3}{(1-y)^2} [x(1-x-y)]^{n+1} \prod_{i=1}^{2n+2} y^{m_i} \quad (9)$$

The subchaim megnetisation per spin in the idrection of the local filed, $m(1)$ is given by

$$m(1) = 1-2\left\{ \sum_{n=1}^{\infty} \sum_{m_i=0}^{\infty} nP(2n,\{m\}) + \sum_{n=0}^{\infty} \sum_{m_i=0}^{\infty} (n+1)P(2n+1,\{m\})\right\} \quad (10)$$

Using (8) and (9), we find

$$\sum_{n=1}^{\infty} \sum_{m_i=0}^{\infty} n\underline{P}(2n,\{m\}) = \frac{2x^3(1-x-y)^3}{(1-y)\{(1-y)^2 - x(1-x-y)^2\}} \quad (11)$$

$$\sum_{n=0}^{\infty} \sum_{m_i=0}^{\infty} (n+1)P(2n+1,\{m\}) = \frac{x(1-x-y)\{x^3 + (1-x-y)^3\}}{\{(1-y)^2 - x(1-x-y)\}^2} \quad (12)$$

Therefore,

$$m(1) = 1-2.\frac{x(1-x-y)\{2x^2(1-x-y)^2 + (1-y)(x^3+(1-x-y)^3)\}}{(1-y)\{(1-y)^2 - x(1-x-y)\}^2} \quad (13)$$

$$\lim_{y \to 0} m(1) = \frac{\{1-x(1-x)\}^2 - 2x(1-x)\{x^3 + (1-x)^3 + 2x^2(1-x)^2\}}{\{1-x(1-x)\}^2}$$

which is the same as that obtained by Williams (1981). The fraction of the frustrated bond is

$$\sum_{n=0}^{\infty} \sum_{mi=0}^{\infty} P(2n,\{m\}) = \frac{2x^2(1-x-y)^2}{(1-y)\{(1-y)^2 - x(1-x-y)\}} \quad (14)$$

Taking the $y=0$ limit in equation (14) gives

$$\sum_{n=0}^{\infty} P(2n,0) = \frac{2x^2(1-x)^2}{(1-x(1-x))},$$

which is the same as that obtained by Williams (1981). From equations (3) and (14) we find

$$E_{bond} = -J + 2J.\frac{2x^2(1-x-y)^2}{(1-y)\{(1-y)^2 - x(1-x-y)\}} \quad (15)$$

Thus, from equations (13) and (15), the ground state energy per spin, $E_o(1)$, is given by

$$E_o(1) = -J + 4J.\left[\frac{x^2(1-x-y)^2}{(1-y)\{(1-y)^2 - x(1-x-y)\}}\right]$$

$$\left\{1 - \frac{2x(1-x-y)\{2x^2(1-x-y)^2 + (1-y)(x^3 + (1-x-y)^3)\}}{(1-y)\{(1-y)^2 - x(1-x-y)\}^2}\right\} B$$

$$, J < B < 2J \qquad (16)$$

The zero point entropy, S_0 *, is given by the logarithm of the ground state degeneracy. Thus*

$$S_o = \sum_{n=0}^{\infty} \sum_{mi=0}^{\infty} P(2n,\{m\}) \log[(m_1+1) + (m_3+1) + (m_5+1) + \ldots + (m_{2n+1}+1)]$$

$$= \frac{2x^2(1-x-y)^2}{(1-y)^2} \sum_{n=0}^{\infty} [x(1-x-y)]^n \sum_{m_1 m_2 m_3 \ldots m_{2n+1} = 0}^{\infty} \Pi_{i=1}^{2n+1} y^{mi}$$

$$\log\left|\left(\sum_{i=0}^{n}(m_{2i}+1) + (n+1)\right|\right.$$

$$= \frac{2x^2(1-x-y)^2}{(1-y)^2} \sum_{n=0}^{\infty} \sum_{N=0}^{\infty} \frac{(N+n)!}{N!n!}\left[\frac{x(1-x-y)}{1-y}\right]^n y^N \log(N+(n+1))$$

$$= \frac{2x^2(1-x-y)^2}{(1-y)^2} \sum_{M=0}^{\infty} \sum_{n=0}^{\infty} \frac{M!}{n!(M-n)!} \frac{x(1-x-y)}{(1-y)}^n y^{M-n} \log(m+1)$$

$$= \frac{2x^2(1-x-y)^2}{(1-y)^2} \sum_{M=0}^{\infty} \left[\frac{x(1-x-y)}{(1-y)} + y\right]^M \log(M+1) \qquad (17)$$

$$\lim_{y\to 0} S_o = 2X^2(1-x)^2 \sum_{M=0} (x(1-x))^M \log(M+1),$$

which is the same as that obtained by Williams (1981).

4. The Model in the Range of Magentic Field $2J/3 < B < J$.

In the range of the field $2J/3 < B < J$*, the idea of superspins and superbonds is applied. The system is now assumed to be composed of superspins and superbonds of order two, while the subchains terminators are of order three or higher. In this range of the magnetic field we are only required to define superspins of order two and three, and superbonds of order two.*

A superspin of order two is defined as two ($\tau_i = 1$ *) spins ortwo (* $\tau_i = -1$ *) spins separated by* m_i *(* $\tau_i = o$ *) spins. Thus, a flipped superspin will be two parallel unsatisfied spins separated by* m_i *(* $\tau_i = 0$ *) spins (a spin is said to be unsatisfied if it is pointing against the direction of its own local field).*

An antiferromagnetic superbond of order two is defined as an even number of($\tau_i \neq 0$ *) successive antiparallel fields separated and terminated by (* $\tau_i = 0$ *) fields. Thus,*

$$P(2\text{-AFSB}) = \sum_{n=0} (P(1\text{-SS}\uparrow))^n . (P(1\text{-SS}\downarrow))^n .$$

$$\sum_{mi=0}^{\infty} (P(1\text{-SS}))^{m_i}\}^{2n+1}$$

$$= -\frac{(1-y)}{\{(1-y)^2 - x(1-x-y)\}}, \qquad (18)$$

where $\tau_i \neq 0$ *represented by* $\{.\}$ *such that* $\tau_i = 1$ *for . and -1 for* $\downarrow$

A ferromagnetic superbond of order two is defined as an antiferromagnetic superbond of order two terminated at one of its ends by a satisfied spin which is parallel to the previous satisfied spin. Therefore,

$$P(2\text{-FSB}\uparrow) = \frac{x(1-y)}{\{(1-y)^2 - x(1-x-y)\}} \qquad (19)$$

Similarly,

$$P(2\text{-FSB}\downarrow) = \frac{(1-x-y)(1-y)}{\{(1-y)^2 - x(1-x-y)\}} \qquad (20)$$

A superspin of order three is defined as a superspin of order two linked to a single satisfied spin by an antiferromagnetic superbond of order two, such that this single satisfied spin is parallel to the superspin i.e. a ferromagnetic superbond of order two. Thus,

$$P(3\text{-SS}\uparrow) = \frac{x^3}{\{(1-y)^2 - x(1-x-y)\}} \qquad (21)$$

Similarly,

$$P(3\text{-SS}\downarrow) = \frac{(1-x-y)^3}{\{(1-y)^2 - x(1-x-y)\}} \qquad (22)$$

The probability factors for subchains composed of 2n and 2n+1 2-SS' (n=0,1,2,.....) and terminated by suprspins of order three are:

$$P(2n+1\uparrow) = \frac{(1-y)x^4}{\{(1-y)^2 - x(1-x-y)\}^2} \cdot \left[\frac{x^2(1-x-y)^2}{\{(1-y)^2 - x(1-x-y)\}^2}\right]^{n+1} \qquad (23)$$

$$P(2n+1\downarrow) = \frac{(1-y)(1-x-y)^4}{\{(1-y)^2 - x(1-x-y)\}^2} \cdot \left[\frac{[x^2(1-x-y)]^2}{\{(1-y)^2 - x(1-x-y)\}^2}\right]^{n+1} \qquad (24)$$

$$P(2n) = \frac{2(1-y)x^3(1-x-y)^3}{\{(1-y)^2 - x(1-x-y)\}^3} \cdot \left[\frac{x^2(1-x-y)^2}{\{(1-y)^2 - x(1-x-y)\}^2}\right]^n$$

Therefore,

$$\sum_{n=0}^{\infty}(n+1)\left[\underline{P}(2n+1\uparrow)] + \underline{P}(2n+1\downarrow)\right] + \sum_{n=1}^{\infty} n\,P(2n)$$

$$= \left[\frac{x^4 + (1-x-y)^4 + 2x^3(1-x-y)^3}{\{(1-y)x^2(1-x-y)\}}\right].$$

$$\frac{(1-y)x^2(1-x-y)^2}{\{((1-y)^2 - x(1-x-y))^2 - x^2(1-x-y)^2\}^2} \quad (25)$$

The above result together with equation (13) gives:

$$m(2) = 1-2\ .\ \frac{x(1-x-y)\{2x^2(1-x-y)^2 + (1-y)(x^3+(1-x-y)^3)\}}{(1-y)\ \{(1-y)^2 - x(1-x-y)\}^2} -$$

$$\frac{4(1-y)x^2(1-x-y)^2}{\{((1-y)^2 - x(1-x-y))^2 - x^2(1-x-y)^2\}^2}\cdot\left[\frac{x^4 + (1-x-y)^4 + 2x^3(1-x-y)^3}{\{(1-y)^2 - x(1-x-y)\}}\right] \quad 26$$

The fraction of the frustrated bond is given by

$$\sum_{n=0}^{\infty} P(2n) = \frac{2x^3(1-x-y)^3}{\{(1-y)^2 - x(1-x-y)\}\ \{(1-y)^2 - 2x(1-x-y)\}(1-y)} \quad (27)$$

Therefore, the ground state energy, $E_o(2)$, is

$$E_o(2) = -J\ 1 - \frac{4x^3(1-x-y)^3}{\{(1-y)^2 - x(1-x-y)\}\ \{(1-y)^2 - 2x(1-x-y)\}\ (1-y)}$$

$$B\ \frac{(1-y)\{(1-y)^2 - x(1-x-y)\}^2 - 2x(1-x-y)\ \{2x^2(1-x-y)^2 + (1-y)(x^3+(2-x-y)^3)\}}{(1-y)\ \{(1-x)2 - x(1-x-y)\}2} -$$

$$\frac{4(1-y)\ x^2(1-x-y)^2}{[\{(1-y)^2 - x(1-x-y)\}^2 - x^2(1-x-y)^2]^2}$$

$$\frac{x^4 + (1-x-y)^4 + 2x^3(1-x-y)^3}{[(1-y)^2 - x(1-x-y)]} \quad (28)$$

5. Generalization of the Model

Let us now examine the general case when $2J < B < 2J$. The system is $r+1$ r considered to be built up of subchains of superspins and superbonds of order r and terminated by superspins of order $(r+1)$ or higher. A general expression for the probability factor of the superspins and superbonds can be deduced by finding their form for some finite values of r, which are

$$P(4\text{-SS}\uparrow) = \frac{x^r}{a_{r-1}}, \qquad P(r\text{-SS}) = \frac{(1-x-y)^r}{a_{r-1}}$$

$$P(r\text{-AFSB}) = \frac{a_{r-1}}{a_r},$$

$$P(r\text{-FSB}\uparrow) = \frac{xa_{r-1}}{ar}, \qquad P(r\text{-FSB}\downarrow) = \frac{(1-x-y)a_{r-1}}{ar} \quad (29)$$

where a^r satisfies the recurrence relation

$$a_r = (1-y)a_{r-1} - x(1-x-y)a_{r-2} \qquad r > 2, \quad (30)$$

with

$$a_0 = 1 \text{ and } a_1 = (1-y).$$

Also a_r has the solution

$$a_r = \sum_{i=0}^{[\frac{r}{2}]} (\,i\,)\,(-1)^i (1-y)^r\ .\ \frac{x(1-x-y)}{(1-y)^2}^{\,i} \quad (31)$$

where $r/2$ is the integer part of $r/2$ and is the binominal coefficient. It can be shown that

$$a_k = \frac{(1-x-y)^{k+1} - x^{k+1}}{(1-x-y) - x}, \quad (32)$$

and

$$a_k^2 - a_{k-1}\,a_{k+1} = x^k (1-x-y)^k, \quad (33)$$

see Doman and Williams (1981). Consider a subchain which has n r-SS's in the interior, terminated at both ends by an (r+1)-SS. Using (29), the probability factors related to the above subchain when n is even an odd are

$$\underline{P}(2n+1) = \frac{a_{r-1}(x^{r+2} + (1-x-y)^{r+2})}{a_r^2}\ .\ \left[\frac{x^r(1-x-y)^r}{a_r^2}\right]^{n+1} \quad (34)$$

by taking into account whether both terminators are up or down.

$$P(2n) = 2x^{r+1}\frac{(1-x-y)^{r+1}a_{r-1}}{a_r^3}\ .\ \frac{x^r(1-x-y)^r}{a_r^3}^{\,n}, \quad (35)$$

by taking into account the two possibilities of the opposite direction terminators. In a section containing 2n r-SS's with $\tau_i \neq 0$ in the interior, at most n r-SS's can change direction in the range of the field $2J/r+1 < B < 2J/r$ leaving one ferromagnetic leaving one ferromagnetic bond frustrated. While, at most n+1 r-SS's can flip relieving all the frustrated ferromagnetic bonds in the sections composed of 2n+1 r-SS's. Thus, the fraction of the frustrated bond, using equation (35), is given by

$$\sum_{n=0} P(2n) = \frac{2x^{r+1}(1-x-y)^{r+1}}{\alpha_r\, a_{r+1}} \quad (36)$$

The ground state energy is given by

$$E_o(r) = E_{bond} - m(r)B, \qquad (37)$$

where $m(r)$ is the magnetisation per spin in the direction of the local field. Hence,

$$E_{bond} = -J + \frac{4x^{r+1}(1-x-y)^{r+1}[(1-x-y)-x]^2}{[(1-x-y)^{2r+3} + x^{2r+3} - (1-y)x^{r+1}(1-x-y)^{r+1}]} \qquad (38)$$

where we have made use of equation (32). Also,

$$\lim_{y \to 0} E_{bond} = -J + 4J \frac{x^{r+1}(1-x)^{r+1}(1-2x)^2}{[(1-x)^{2r+3} + x^{2r+3} - x^{r+1}(1-x)^{r+1}]}$$

which is the same as that obtained by Williams (1981) in his random field problem. Using the probability factors given in equations (34) and (35), the number of flipped spins is given by

$$r.\sum_{n=0}^{\infty}(n+1)\,\underline{P}(2n+m_i) + r.\sum_{n=1}^{\infty} n\,\underline{P}(2n,m_i) =$$

$$r\ .\ \frac{x^r(1-x-y)^r a_{r-1}}{[a_r^2 - x^r(1-x-y)^r]^2}\ .\left[x^{r+2} + (1-x-y)^{r+2} + \frac{2x^{r+1}(1-x-y)^{r+1}}{a_r}\right] \qquad (39)$$

Thus, a general form for the magnetisation per spin in the direction of the local field can be obtained from the following recurrence relation

$$m(r) = m(r-1) - \frac{2ra_{r-1}x^r(1-x-y)^r}{[a_r^2 - x^r(1-x-y)^r]^2}\ .\left|x^{r+2} + (1-x-y)^{r+2} + \frac{2x^{r+1}(1-x-y)^{r+1}}{a_r}\right| \qquad (40)$$

A substitution of the results given by equations (40) and (38) in euqation (37) produced the general form for the ground state energy which is

$$Eo(r) = -J + 4J\frac{x^{r+1}(1-x-y)^{r+1}[(1-x-y)-x]^2}{[(1-x-y)^{2r+3} + x^{2r+3} - (1-y)x^{r+1}(1-x-y)^{r+1}]} - m(r)\,B,$$

$$2J/(r+1) < B < 2J/r \qquad (41)$$

6. The Zero-Point Entropy

In this section we establish a general form for the ground state entropy for the external magnetic field within the range $2J/(r+1)$ $r < B < 2J/r$. Consider the probability factor for an antiferromagnetic superbond of order r and having a degeneracy k is represented by $\underline{P}^r(k)$. Let us start with the evaluation of the probability factor corresponding to some simple configurations in which $P(1\text{-}FB) = 1$ and $P(1\text{-}(\tau=0))$ is the probability factor for a single ($\tau=0$) spin.

(i) $\underline{P}^r(1) = 2\underline{P}(r+1\text{-}SS\uparrow)\ .\ \underline{P}(r+1\text{-}SS\downarrow)\ .\ \underline{P}(1\text{-}FB)$

(ii) $\underline{P}^r(2) = 2\underline{P}(r+1\text{-}SS\uparrow)\ .\ \underline{P}(r+1\text{-}SS\downarrow)\ .\ \{P(1\text{-}(\tau=0)SS)\ .\ \underline{P}(1\text{-}FB) + \sum_{j=1}^{r}\underline{P}(j\text{-}SS\uparrow)\ .\ P(j\text{-}AFSB)\ .\ P(j\text{-}SS\downarrow)\ .\ P(1\text{-}FB)$

(iii) $P^r(3) = 2P(r+1\text{-}SS\uparrow)\ .\ P(r+1\text{-}SS\downarrow)\ .\ P(1\text{-}FB)\ .\left[P(1\text{-}(\tau=0)SS)\ .\ \underline{P}(1\text{-}FB) + \sum_{j=1}^{r}\underline{P}(j\text{-}SS\uparrow)\ .\ \underline{P}(j\text{-}AFSB)\ .\ \underline{P}(j\text{-}SS\downarrow)\ .\ P(1\text{-}FB)\right]^2$

Thus, in general, we may write $\underline{P}^r(k)$ as

$$\underline{P}^r(k) = 2\underline{P}(r+1\text{-}SS\uparrow)\ .\ P(r+1\text{-}SS\downarrow)\ .\left\{P(1\text{-}(\tau=0)SS)\ .\ P(1\text{-}FB) + \sum_{j=1}^{r}P(j\text{-}SS\uparrow)\ .\ P(j\text{-}AFSB)\ .\ P(j\text{-}SS\downarrow)\ .\ P(1\text{-}FB)\right\}^{k-1}$$

$$= \frac{2x^{r+1}(1-x-y)^{r+1}}{a_r^2}\left[y + \sum_{j=1}^{r}\frac{x^j(1-x-y)^j}{a_{j-1}a_j}\right]^{k-1} \qquad (42)$$

The summation over j can be performed to give

$$\sum_{j=1}^{r}\frac{x^j(1-x-y)^j}{a_{j-1}a_j} = \frac{x(1-x-y)a_{r-1}}{a_r} \qquad (43)$$

which may be proved using induction. Hence,

$$P^r(k) = \frac{2x^{r+1}(1-x-y)^{r+1}}{a_r^2}\left[y + \frac{x(1-x-y)a_{r-1}}{a_r}\right]^{k-1} \qquad (44)$$

Making use of the recurrence relation (30), the probability equation (44) may be written as

$$P^r(k) = \frac{2x^{r+1}(1-x-y)^{r+1}}{a_r^2}\ .\ \left[\frac{a_r - a_{r+1}}{a_r}\right]^{k-1} \qquad (45)$$

The zero-point entropy, $So(r)$, can be obtained by making use of equation (45) as follows

$$S_o(r) = \frac{2x^{r+1}(1-x-y)^{r+1}}{a_r}\ .\ \sum_{k=0}^{\infty}\left[\frac{a_r - a_{r+1}}{a_r}\right]^k \log(k+1)$$

The magnetisation per spin in the direction of the local field and the zero-point entropy are represented by figures (a) to (d). All figures show steps in the magnetisation per spin in the direction of the local field and the zero-point entropy at the critical values of the field. These steps occur as a result of the superspins flipping as the external magnetic field reduced and passes through its critical values at $B = 2J/r$

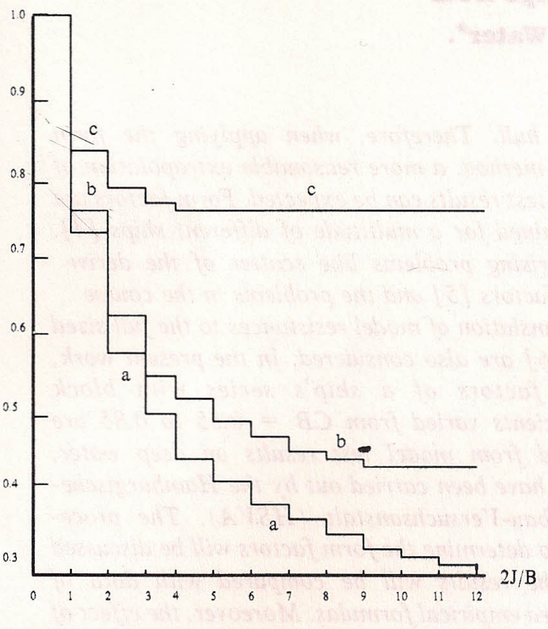

Fig (5-a) The partial net magnetization as a function of 2J/B for differenct concentrations (x) at T=0 and y=0.15. (a) x = 0.25, (b) x = 0.5, (c) x = 0.75

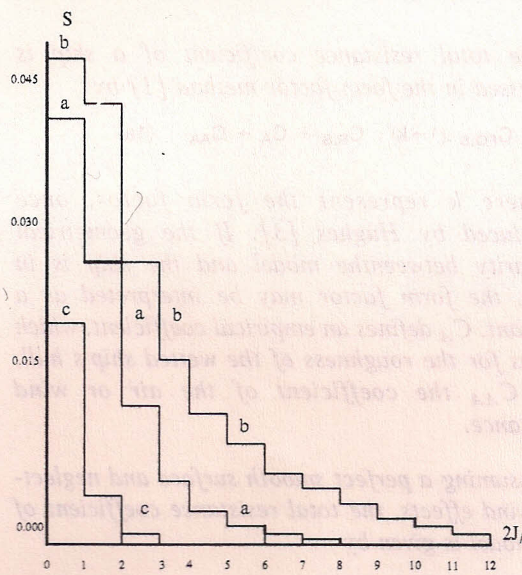

Fig (5-b) Entropy as a function of 2J/B for different concentrations (x) at T=0 and y=0.15 (a) x = 0.25, (b) x = 0.5 , (c) x = 0.75

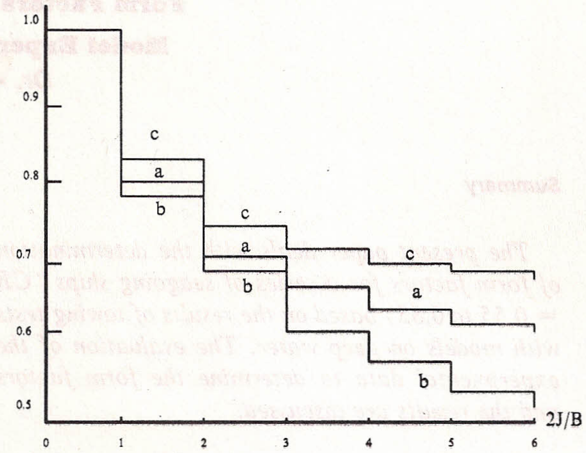

Fig (5-c) The partial net magnetization as a function of 2J/B for different concentrations (x) at T=0 and y = 0.35 (a) x = 0.2 , (b) x = 0.35 , (c) x = 0.5

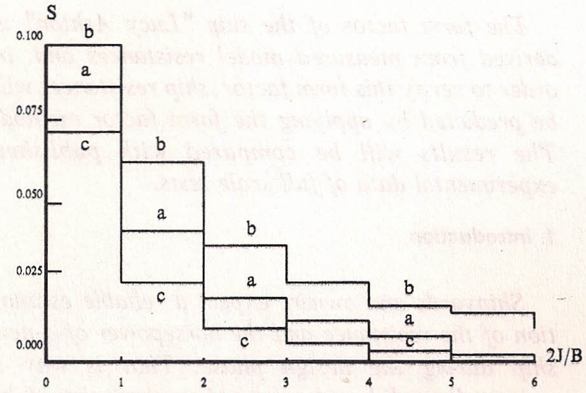

Fig (5-d) Entropy as a function of 2J/B for different concentrations (x) at T=0 and y = 0.35.

(a) x = 0.2 , (b) x = 0.35 (c) x = 0.5

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