JOURNAL OF ENGINEERING RESEARCH

V.4 June 1995

This Journal is Published by the engineering research center - College of engineering - El-fateh University Tripoli -LIBYA

Contents

FINITE ELEMENT METHOD ; APPLICATION TO HEAT CONDUCTION PROBLEMS

M. T. Abujelala. Assistant Prof. of Aero. Engineering

University of Alfateh, Tripoli, Libya

Summary

The application of finite element technique to problems involving steady state and transient heat conduction within a medium is the objective of this paper. The flexibility of this technique to the problems of irregular geometry is demonstrated by using a triangular element. Spacewise discretization of the governing equation along with three types of boundary conditions was performed using the variation principles. The resulting form of the control equation is computerized. The code is capable to divide the cited domain into triangles once its dimensions and step values are provided. The results show that the finite element scheme used is stable and the technique of finite element provides numerical solution of excellent agreement with the exact values for the considered cases. However, the solution required high capacity computer and long execution time as compared with other techniques such as finite difference.

ii. Nomenclature:

C	Specific heat
$[C]$	Capacitance matrix
h	Convective heat transfer coefficient
K_x, K_y	Thermal conductivity in x - y directions
$[K]$	Conductivity matrix q Boundary heat flux
Q	Heat flow
S	Surface boundary
T	Temprature vector
$\bar{T}$	Surrounding temperature
T_s	Boundary specified temperature
t	Time
x,y	Cartisian coordinates
ζ,ξ	Isoparametric coordinates
ρ	desity

1. Introduction:

The availability of high speed and capacity computers have led to more accurate and dependable numerical solution to many engineering complex problems. The governing differential equations for medium with irregular geometry have been well defined numerically using finite element technique. This technique is based on the subdivision of the region of interest into a number of subregions and the cited problem is solved on these subregions and the solution is then proceed to include all the region. The underline ideas of finite element were first discussed in References 1 and 2 and were applied to structural problems by Reference 3. Application of finite element to nonstructural problems, such as fluid flow, was initiated by Reference 4, and to various problems of heat transfer by References 5 and 6. Reference 7 extended the application of the finite element to transient heat conduction in solids with non-linear boundary conditions, i.e. the surface boundary is allowed to vary with time step. Comparison between three numerical integration schemes that may be used in finite element solution were presented in Reference 8. He suggested a relation, which was derived from Galerkin process, for the treatment of fast varying boundary conditions. Computer code for solving non-linear steady state and transient thermal processes was described in Reference 9. An implicit Crank-Nichilson time-integration scheme is used, with consistent or lamped capacitance matriles as an option, by Reference 10. Time is treated by Reference 11 as an additional dimension in the solution domain of treansient heat problems; heat transfer by convection, conduction and radiation were considered.

2. Objectives:

The purpose of this paper is to deepen the existing understanding of the application of the finite element analysis to heat conduction problems exposed to different boundary conditions. Also, some of the finding of previous relevant investigations for two dimensional transient and steady heat conduction are computerized. The code is capable to divide the domain into triangles once its dimensions are provided. Three boundary conditions can be imposed on the domain. These are; specified surface temperature, isolated and convected or conducted boundary conditions.

3. Mathematical Background

The governing equation for two dimensional unsteady heat conduction problems is quasi-liner parapolic type. This equation is in the form;

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial T}{\partial y}\right) + Q - \rho C \frac{\partial T}{\partial t} = 0 \qquad [1]$$

The solution of this equation depends on the type of the imposed boundary and the initial conditions. Spacewise discretization of Eq.[1] subjected to the boundary conditions;

$$K_x \frac{\partial T}{\partial x} + K_y \frac{\partial T}{\partial y} + q = 0 \qquad \text{on } S_1$$

$$K_x \frac{\partial T}{\partial x} + K_y \frac{\partial T}{\partial y} + h\,(T - T_\infty) = 0 \qquad \text{on } S_2$$

$$T = T_s \qquad \text{on } S_3,$$

may be performed using the Galerkin method (12) or the variation principle (13). The variation principle is based on the selection of a function F(x) such that any small arbitrary change, δF(x) will not change the integral of F(x). The minimization form of this function is given by:

$$\delta I_V = \int_V \left[\frac{\partial F}{\partial T} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial T_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial T_y}\right) \right] \delta T \, dV +$$

$$\int_S \left[L_x \frac{\partial F}{\partial T_x} + L_y \frac{\partial F}{\partial T_y} \right] \delta T \, ds$$

where

$T_x = \frac{\partial T}{\partial x}$, $T_y = \frac{\partial T}{\partial y}$, *V is the control volume,*

and s represents the surface area of the control volume. Comparison between Eqs [1] and [2] yields the following relations:

$\frac{\partial F}{\partial T} = \left(\rho c \frac{\partial T}{\partial t} - Q\right)$, and this for $Q \neq F(T)$ gives

$$F = \left(\rho C \frac{\partial T}{\partial t} - Q\right).\,T$$

$$\frac{\partial F}{\partial T_x} = K_x \frac{\partial T}{\partial x} \qquad \text{or} \qquad F = \frac{1}{2} K_x \left(\frac{\partial T}{\partial x}\right)^2$$

$$\frac{\partial F}{\partial T_y} = K_y \frac{\partial T}{\partial y} \qquad \text{or} \qquad F = \frac{1}{2} K_y \left(\frac{\partial T}{\partial y}\right)^2$$

Thus the integral of the required function is represented by :

$$I_V = \frac{1}{2} \int_V \left[K_x \left(\frac{\partial T}{\partial x}\right)^2 + K_y \left(\frac{\partial T}{\partial y}\right)^2 + 2, \left(\rho C \frac{\partial T}{\partial t} - Q\right).T \right] dV \qquad [3]$$

The surface integral depends on the boundary conditions. Generally the boundary may be represented by constant values or/and by;

$$K_x \frac{\partial T}{\partial x} + K_y \frac{\partial T}{\partial y} + q + h\,(T - T_\infty) = 0$$

and the function representation corresponding to these boundaries;

$$I_s = \int_S \left(q\,T + \frac{h}{2}(T - T_\infty)^2\right) dS \qquad [4]$$

4. Finite Element Formulation:

The triangular element, shown, is selected for this investigation to help the simulation of curved boundaries to a high degree of accuracy. This element is based upon isoparametric formulation (12), and the temperature within the element is given by:

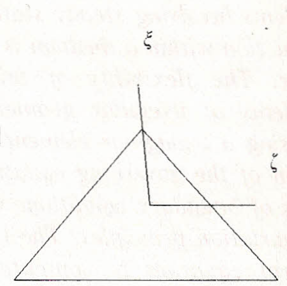
$$T(\zeta, \xi) = \sum_{i=1}^{3} N_i T_i$$

where N is the shape function and it is in the form:

$$N_1 = \frac{1}{4}\left[(1-\zeta).(1-\xi)\right]$$

$$N_2 = \frac{1}{4}\left[(1+\zeta).(1-\xi)\right]$$

$$N_3 = \frac{1}{2}\left[1+\xi\right]$$



Equations [3] and [4] are transformed into ζ - ξ co-ordinates by the following relations:

$$\begin{bmatrix} \frac{\partial N}{\partial \zeta} \\ \frac{\partial N}{\partial \xi} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \end{bmatrix} \begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{bmatrix}$$

The nodal temperature (φ) is related to the element temperature by :

$T = [N]\{\phi\}$, and

$$\frac{\partial T}{\partial t} = [N] \frac{\partial}{\partial t}\{\phi\}$$

Implementing the above formulation into Eqs.[3] and [4] yields the following results:

$$I_v = \frac{1}{2} \int \left[\{\phi\}^T [B][K]\{\phi\} + \rho c \{N\} \frac{\partial}{\partial t}\{\phi\} - Q\{N\}^T\{\phi\} \right] dV$$

$$I_s = q \int_V \{N\}\{\phi\}\, dS + \frac{h}{2} \int_s \left(\{N\}\{\phi\} - T_\infty \{N\}\right)^2 ds$$

$$\text{where } [B] = \begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{bmatrix}^T \quad \text{and } [K] = \begin{bmatrix} K_x & 0 \\ 0 & K_y \end{bmatrix}$$

Minimization of Iv and Is with respect to {φ} gives;

$$[K]\{\phi\} + [c] \frac{\partial}{\partial t}\{\phi\} + \{F\} = 0 \qquad [5]$$

such that:

$$[K] = \int_V [B]^T [K][B]\, dV + h \int_S \{N\}^T \{N\}\, ds$$

$$[c] = \rho c \int_V \{N\} \{N\} dV$$

$$[F] = -Q \int_V \{N\}^T dV + q \int_S \{N\}^T dS - h T_\infty \int_S \{N\}^T dS$$

The global matrix is obtained by summing up all the element contributions. The evaluation of integrals required by [5] may be done using any numerical technique, and two point gaussian quadrature is selected for this study.

5. Transient Algorithm

The solution of [5] in time domain can be achieved by more than one technique. Reference (14) assumed that the temperature varies linearly in the small interval between $t-\Delta t$ and $t+\Delta t$ and he approximated [5] by :

$$\{T\}_{t+\Delta t} = -\left([K] + \frac{3}{2\Delta t}[c]\right)^{-1}_t \left(3\{F\}_t + \frac{3}{2\Delta t}[c]\{T\}_{t-\Delta t} - \right.$$

$$[K]_t = (\{T\}_t + \{T\}_{t-\Delta t})) \qquad [6]$$

He reported that this algorithm is unconditionally stable and converge in the context of finite element formulation. However, when this explicit algorithm was implemented in the generated code did not provide an acceptable results. Reference (9) suggested to use an implicit time integration scheme. In this scheme, the solution at each time step is computed at the middle of the time interval t, and accordingly Eq.[5] is re-written as :

$$\{T\}_a = -\left([K] + \frac{2}{\Delta t}[c]\right)^{-1}_t \left(\frac{2}{\Delta t}[C]\{T\}_t + \frac{1}{2}(\{F\}_t + \{F\}_{t+\Delta t})\right)$$

and $\{T\}_{t+\Delta t} = 2\{T\}_a - \{T\}_t$

this algorithm is known as Crank-Niulson scheme, and it is unconditionally stable if the time step is well selected. An explicit-implicit iterative scheme was adopted by Reference (5). It is a combination of the two previous schemes. This scheme, as well as previous discussed time schemes, have a common problem with the selection of time step value. Reference (16) explores this problem in details and suggested a critical time step defined by:

$$t_c = \mathrm{Min}\left[\frac{C_{ii}}{K_{ii} + \sum_{j\neq i=1}^{N} |K_{ij}|}\right]_{i=1}^{N}$$

where N is the number of considered domain elements. This suggestion was implemented in the computer code provided by this study however, the critical time step seems very small, where time step of ten times greater has provided very close answer to the exact solution. The scheme adopted by this paper is based on the relation provided by Reference (17), thus the temperature at time t and at $t+\Delta t$ have the relation,

$$\{T\}_{t+\Delta t} = \{T\}_t + (1-\beta)\{T\}_t + \{T\}_t + \beta\{T\}_{t+\Delta t}$$

with $\beta = \frac{2}{3}$, and Eq. [5] is reduced to :

$$\left(\frac{2}{3\Delta t}[c] + [K]\right)\{T\}_{t+\Delta t} = \left(\frac{2}{3\Delta t}[c] - \frac{1}{2}[K]\right)\{T\}_t \left(-\frac{1}{2}[F]_t + [F]_{t+\Delta t}\right)$$

6. Result and Discussion

To verify the computer code results, cases of known exact solutions are solved by the code. This includes the steady state form of Eq.[5] with the following boundary conditions;

$T = 100 \sin(\pi x/0.6)$ on top side

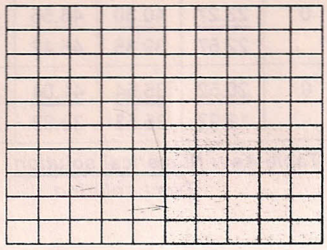
$\frac{\partial T}{\partial x} = 20h$ on bottom side

$T_s = o$ on other sides

The solution of this case over an element of dimension 0.7 x 0.3 compared to the exact solution is presented inTable 1-a. Table 2-b provides the solution for the same problem but with isolated bottom boundary, i.e.,

$\frac{\partial T}{\partial x} = 0$

The accuracy and flexibility of the code were demonstrated by solving transient heat transfer in the shown domain. The boundary and the initial conditions used are:



$T_O = 30$

$T_t = 0$ on all the boundaries, where suffix o and t refer for initial and at time t, respectively.

Due to the symmetry of the domain around its centre, the solution can be obtained by using the hatched section of the domain with the following boundries:

$\frac{\partial T}{\partial x} = 0$ on the right and top sides of the domain.

$\frac{\partial T}{\partial t} = 0$ on the other sides

The solution of these two cases, over a domain 3x3 with equal intervals is given in Table 2. Thermal conductivity and specific heat by density were given the values of 1.25 Btu/(hr.m.F) and 1.0 Btu/(m^3.F), respectively. The time step was given the value of 0.04 hr.

The results show that the finite element scheme is stable and the technique of finite element provides numerical solution of excellent agreement with the exact values for the considered cases. Also, comparison of data of Table 2-a and 2-b reveals that the simple the boundary conditions the closer is the solution to the exact values. This conclusion was drawn as a result of comparing the analytical solution with the numerical values presented in Tables 1 & 2.

Table 1. Steady State Heat Conduction@

a) Convected Boundary

0	49.91	86.23	100.00	86.23	49.91	0
0	28.35 / 28.02	49.07 / 48.85	56.75 / 56.39	49.08 / 50.56	28.36 / 28.02	0
0	14.42 / 14.31	24.97 / 24.78	28.85 / 28.60	24.97 / 24.78	14.43 / 14.31	0
0	4.35 / 4.34	7.53 / 7.66	8.70 / 8.85	7.53 / 7.66	4.41 / 4.43	0

b) Isolated Boundary

0	49.91	86.23	100.00	86.23	49.91	0
0	32.25 / 31.89	55.83 / 55.23	64.55 / 63.76	55.83 / 55.23	32.25 / 31.89	0
0	23.27 / 22.67	40.30 / 39.35	46.55 / 45.42	40.30 / 39.32	23.27 / 22.67	0
0	20.52 / 19.92	35.54 / 35.53	41.04 / 39.83	35.54 / 35.53	20.52 / 19.92	0

@ Table Key: Numerical solution / Exact solution

Table 2. Transient Heat Conduction

a) Specified Boundares

0	0	0	0	0	0	0	0	0	0	0
0	0.173 / 0.173	0.329 / 0.329	0.454 / 0.453	0.537 / 0.533	0.562 / 0.560	0.535 / 0.533	0.456 / 0.453	0.332 / 0.329	0.175 / 0.173	0
0	0.329 / 0.329	0.627 / 0.626	0.863 / 0.862	1.016 / 1.013	1.070 / 1.065	1.018 / 1.013	0.867 / 0.862	0.630 / 0.626	0.332 / 0.626	0
0	0.454 / 0.453	0.864 / 0.862	1.190 / 1.186	1.399 / 1.394	1.472 / 1.466	1.401 / 1.394	1.192 / 1.186	0.867 / 0.862	0.456 / 0.453	0
0	0.534 / 0.533	1.016 / 1.013	1.399 / 1.394	1.645 / 1.639	1.731 / 1.723	1.646 / 1.639	1.401 / 1.394	1.018 / 1.013	0.535 / 0.533	0
0	0.562 / 0.560	1.070 / 1.065	1.472 / 1.065	1.731 / 1.723	1.820 / 1.812	1.731 / 1.723	1.472 / 1.466	1.070 / 1.065	0.562 / 0.562	0
0	0.535 / 0.533	1.018 / 1.013	1.401 / 1.394	1.646 / 1.639	1.731 / 1.723	1.645 / 1.639	1.390 / 1.394	1.016 / 1.013	0.534 / 0.533	0
0	0.456 / 0.453	0.867 / 0.862	1.192 / 1.186	1.401 / 1.394	1.472 / 1.466	1.399 / 1.394	1.190 / 1.186	0.864 / 0.862	0.454 / 0.453	0
0	0.332 / 0.329	0.630 / 0.626	0.867 / 0.862	1.018 / 1.013	1.070 / 1.065	1.016 / 1.013	0.864 / 0.862	0.627 / 0.626	0.329 / 0.329	0
0	0.175 / 0.173	0.332 / 0.329	0.456 / 0.453	0.535 / 0.533	0.562 / 0.560	0.534 / 0.533	0.454 / 0.453	0.329 / 0.329	0.173 / 0.173	0
0	0	0	0	0	0	0	0	0	0	0

b) Isolated & Specified Boundaries

0	0	0	0	0	0	0
0	1.864 / 1.812	1.738 / 1.723	1.522 / 1.466	1.108 / 1.065	0.583 / 0.562	0
0	1.724 / 1.723	1.647 / 1.639	1.406 / 1.394	1.024 / 1.013	0.539 / 0.533	0
0	1.438 / 1.466	1.374 / 1.394	1.172 / 1.186	0.854 / 0.862	0.449 / 0.453	0
0	1.032 / 1.065	0.986 / 1.013	0.841 / 0.862	0.612 / 0.626	0.332 / 0.329	0
0	0.539 / 0.560	0.514 / 0.533	0.439 / 0.453	0.319 / 0.329	0.168 / 0.173	0
0	0	0	0	0	0	0

◆ Temperature Distribution at Time = 1.2 hr

7. References Used

1. Chung, T.J., "Finite Element Analysis in Fluid Dynamics", McGraw-Hill, Inc., New York, 1978.

2. Courant, R., "Variational Method for the Solution of Problems of Equilibrium and Vibrations", Bulletin of American Matehmatical Science, Vol.49, 1943, pp.1-23.

3. Turner, M.J.,etal, "Stifness and Deflection Analysis of Complex Structure", Journal of Aeronautical Science, Vol.23, No.9, 1956, pp. 805- 823.

4. Zienkiewicz, O.C.,etal, "Solution of An Isotropic Seepage Problems by Finite Elements", Proceeding of the ASCE, Vol. 92, EMI, 1966, pp. 11-12.

5. Wilson, E.L. and Nickell R.E., "Application of The Finite Element Method to Heat Conduction Analysis" ,Nuclear Engineering and Design, Vol.4, 1967, pp. 276-286.

6. Zienkiewicz, O.C. and Cheung, Y.K., "The Finite Element Method in Structure and Continuum Mechanics", McGraw-Hill, New York 1977.

7. Beckett, R.E. and Chu, S.C., "Finite Element Method Applied to Heat conduction in solids with Non-linear boundary Conditions", Transaction of the ASME, 1973, pp.126-128.

8. Donea, J., "ON The Accuracy of Finite Element Solutions to The Transient Heat Conduction Equation", International Journal for Numerical Method in Engineering, Vol.8, 1974, pp.103-110.

9. Thornton, E.A., "TAP2: A Finite Element Program for Conduction-Forced Convection Thermal Analysis, NASA CR-159038, Dec. 1979.

10. Hsu, M.B. and Nickell, R.E., "Coupled Convective and Conductive Heat Transfer by Finite Element Methods in Flow Problems, edited by Oden, J.T., etal., UAH Press, University of Albama, Huntsville, Ala, 1974, pp.427-449.

11. Chung, K.S., "The Fourt-Dimension Concept in the Finite Element Analysis of Transient Heat Transfer Problems", International Journal of Numerical Method in Engineering, 1981, pp.315-320.

12. Cook, R. D., "Concepts and Application of Finite Element Analysis", John Wiley & Sons, New York, 1980.

13. Segerlind, L.J., "Applied Finite Element Analysis", John Wiley & Sons, New York, 1976.

14. Comini, G., etal, "Finite Element Solution of Non-linear Heat Conduction Problems with Special Reference to Phase Change", International Journal of Numerical Method in Engineering, Vol. 8, 1974, pp.613624.

15. Chin, H.J., "Finite Element Analysis for Conduction and Ablation Moving Boundary", Heat transfer and Thermal Control, edited by Grosbie, A. L., American Institute of Aero. & Astro., New York, 1980.

16. Myers, G.E., "The Critical Time Step for Finite Solutions to Two Dimensional Heat Conduction Transients", Transaction of the ASME, vol.100, 1978, pp. 120-129.

17. MacNeal, R. H., etal, "The NASTRAN Theoretical Manual (level 16.0), NASA-SP- 22(03), March , 1976 (N79-27531, N.T.I.S.)