JOURNAL OF ENGINEERING RESEARCH

V.2 DECEMBER 1991

The Journal of Engineering Research is Published by the Engineering Research Center Tripoli Jamahirya

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تحليل الصفائح الموضوعة على أساسات مطاطية

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ANALYSIS OF RECTANGULAR PLATES WITH ARBITRARY BOUNDARY CONDITIONS RESTING ON ELASTIC FOUNDATIONS

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ABSTRACT

In this paper the analysis of rectangular platès on elastic foundation with arbitrary boundary conditions are considered. The solution approach consists of choosing a series of functions which, term by term, satisfy the governing equation and the boundary conditions on displacement. The boundary condition on slope is satisfied by minimizing the weighted residual procedure.

The closed form solution developed in this paper can be generalized to consider any combination of the boundary conditions.

هذه الورقة تحتوى على تحليل الصفائح الموضوعة على أساسات مطاطية وشروطها الحدودية اختيارية. طريقة الحل التى اتبعت فى هذه الورقة تسمى التحليل بطريقة المسلسلات اللانهائية، وباتخاذ عدد كبير من المعاملات لهذه المسلسلات التى كل منها يحقق المعادلة الرئيسية وجمعها جمعاً جبرياً وجد فى بحوث سبق نشرها، ان هذه الطريقة تعطى حلاً مقارباً للحل الصحيح، حيث أن الشروط الحدودية للإزاحة قد حققت تحقيقاً صحيحاً. أما الشروط الحدودية الميلية فقد حققت باستخدام طريقة تقليل الأخطاء الموزونة.

الطريقة المقدمة فى هذه الورقة يمكن جعلها طريقة عامة بحيث تستعمل فى حل أى صفيحة بأى شروط حدودية.

Introduction:

The plate equation is involving more than one independent variable which may be solved by two different methods.

One method is called a closed form solution, which has been found for a very limited number of cases.

The first closed-form solutions was found by Navier (1) in 1820. Levs (1) found more general solution for rectangular plates with two parallel sides being simply supported and the other two edges having any boundary conditions.

Unfortunately, the analysis of plates whose four edges are free or clamped becomes more complex and the closed-form solution becomes more difficult.

The difficulty is in how to find a suitable deflection function which satisfy the stressed boundary conditions. The main objective of this paper is to find a closed form solution to a rectangular plate resting on elastic foundation with arbitrary conditions.

The other method of solution to the plate equation involves a numerical method such as a finite element and a finite difference method... etc.

The finite element method can be applied to solve plate problems which have any shape and any boundary conditions

The method used in this paper is called a series solution method, which has been applied to solve different type of engineering problems (2 - 5).

Hutchinson (6) and Taylor (7) greatly contributed to this method by using it to solve bending and buckling plate problems respectively.

The series solution method formed as the sum of several basic solutions which satisfy the governing differential equation.

These basic solutions are grouped into series form in such a way that the displacement boundary conditions are satisfied by orthogonalization on the boundary.

The choice of which boundary condition is to be satisfied identically and which is to be satisfied by orthogolization is arbitrary.

Formulation:

The usual assumption in the development of the governing equation of plate resting elastic foundation is that the intensity of the reaction at any point of the botton plate is proportional to the deflection w at that point, if k being the modulus of the foundation, then the differential equation for the deflection curve in rectangular coordinates, becomes:

$$\frac{\delta^4 W}{\delta x^4} + \frac{2\delta^4 W}{\delta x^2\,\delta y^2} + \frac{\delta^4 W}{\delta y^4} = \frac{q}{D} - \frac{k}{D} W \qquad (1)$$

where all the notations are given in the appendix. The solution of the above equation can be expressed in case of simply supported plate as:

$$W(X,Y) = W_p + W_H \qquad (2)$$

Where W_P is the particular solution which can be taken as a function of X or as a function of Y, and due to symmetry of the plate in y-direction as show in fig.1, the homogeneous solution W_H is taken as an even function of Y.

The general solution of eq.(1) for a simply supported plates can be expressed in the following form:

$$W(X,Y) = \sum_{m=1,3,5}^{\infty} \sin \mu_m \times \left(\frac{4q}{\pi D}\,\frac{1}{m(\mu_m^4 + k/D)}\right)$$

$$+ A_m \cosh(\gamma_m y)\cos(\gamma_m y) + B_m \sin h(\beta_m y) \sin(\gamma_m y)$$

The boundary conditions for simply supported plates in x−direction are

$$W(a,Y) = 0$$

$$\frac{\delta^2 W(a,Y)}{\delta X^2} = 0$$

which can be satisfied by choosing

$\mu_m = \dfrac{m\pi}{a}$, and the boundary condition in y−direction are

$$W(x\ ,\ b/2) = 0$$

$$\frac{\delta^2 w(x,b/2)}{\delta y^2} = 0$$

which can be satisfied by choosing

$$A_m = \frac{-G_2}{\sin h\,\phi_m \sin\Psi_m + (\cos h\,\phi_m \cos\Psi_m)^2}\;\frac{\beta_m^2 - \gamma_m^2}{2\gamma_m \beta_m} + \frac{\cos\Psi_m \cos h\,\phi_m}{\sin h_m \sin} \qquad (4)$$

$$B_m = \frac{-\cos h\;\phi_m \cos\Psi_m}{\sin h\;\phi_m \sin\Psi_m} A_m - \frac{G_2}{\sin h\;\phi_m \sin\Psi_m} \qquad (5)$$

$$G_2 = 4q/(mD\pi\;(\mu_m^4 + k/D)) \qquad (6)$$

The solution of simply supported plates represented into new coordinate system as shown in fig.(2) can be written as

$$W(x,y) = -\sum_{m=1,3,5}^{\infty} (-1)^{(m+1)/2} \cos(\mu x)\Big[G_2 + A_m \cos h(\beta_m y)$$

$$\cos(\gamma_m y) + B_m \sin h(\beta_m y) \sin(\gamma_m y)\Big] \qquad [7]$$

Where A_m, B_m and G_2 are as given in eqs.4,5 and 6 respectively.

The general solution of the plate equation on arbitrary boundary condition can be expressed in the following general form:

$$w(x,y) = \begin{Bmatrix} W_p(x) \\ W_p(y) \end{Bmatrix} + s_1 + s_2 \qquad (8)$$

where

$$W_p(x) = -\sum_{m=1,3,5}^{\infty} (-1)^{(m+1)/2} \cos(\eta_m y)\Big[G_1 + G_m \cos h(\delta_m x)$$

$$\cos(\alpha_m x) + D_m \sin h(\delta_m x) \sin(\alpha_m x)\Big] \qquad (9)$$

and

$$Wp(y) = -\sum_{m=1,3,5}^{\infty} (-1)^{(m+1)/2} \cos(\mu_m y)\Big[G_2 + A_m \cos h(\beta_m y)$$

$$\cos(\gamma_m y) + B_m \sin h(\beta_m y) \sin(\gamma_m y)\Big] \qquad (10)$$

Where A_m and B_m and G_2 are as given in eqs.4,5, and 6 respectively, and

$$C_m = \frac{-G_1}{\sin h\,\rho_m \sin\theta_m + (\cos h\,\rho_m \cos\theta_m)^2}\;\frac{\delta_m^2 - \alpha_m^2}{2\alpha_m \delta_m}$$

$$+ \frac{\cos h\,\rho_m \cos\theta_m}{\sin h\,\rho_m \sin\theta_m} \qquad (12)$$

$$D_m = -\frac{\cos h\,\rho_m \cos\theta_m\, C_m}{\sin h\,\rho_m \sin\theta_m} - \frac{G_1}{\sin h\,\rho_m \sin\theta_m} \qquad (13)$$

$$S_1 = \sum_{m=1,3,4}^{\infty} E_m \left[\frac{\sin h(\beta_m y)\sin(\gamma_m y)}{\sin h\,\phi_m \sin\Psi_m}\right.$$

$$\left. - \frac{\cos h(\beta_m y)\cos(\gamma_m y)}{\cos h\,\phi_m \cos\Psi m}\right] \cos(\mu_m x) \qquad (14)$$

$$S_2 = \sum_{m=1,3,5}^{\infty} F \left[\frac{\sin h(\delta_m x)\sin(\alpha_m x)}{\sin h\,\rho_m \sin\theta_m}\right.$$

$$\left. - \frac{\cos h(\delta_m x)\cos(\alpha_m x)}{\cos h\,\rho_m \sin\theta_m}\right] \cos(\eta_m y) \qquad (15)$$

Application to solution of clamped plates

In the following process of a solution the clamped plates from all sides will be considered as an example to this method of solution which can be applied to any arbitrary boundary conditions.

The boundary conditions for plates clamped from all sides can be written as

$$W(a/2\;,\;y) = 0 \qquad (16)$$

$$W(x\;,\;b/2) = 0 \qquad (17)$$

$$\frac{\partial W}{\partial x}(a/2,y) = 0 \qquad (18)$$

$$\frac{\partial W}{\partial y}(x,b/2) = 0 \qquad (19)$$

The equation (16) is satisfied by choosing:

$$\mu_m = \frac{m\pi}{a}$$

and the equation (17) is satisfied by choosing:

$$\eta_m = \frac{m\pi}{b}$$

The remaining boundary conditions are satisfied by minizing the weighted residual at the boundary i.e. by using the orthogonality process.

The equation (18) is approximately satisfied as follows:

$$\int_0^{b/2} \frac{\partial w(a/2,y)}{\partial x} \cos\eta_n y\; dy = 0 \qquad (20)$$

which leads to

$$x + E_m e_{mn} + F_m f_m = 0 \qquad (21)$$

and the equation (19) is approximately satisfied as follows:

$$\int_0^{a/2} \frac{\partial w(x,b/2)}{\partial y} \cos \mu_n x \, dx = 0 \qquad (22)$$

$$y + E_m e_m + F_m f_{mn} = 0 \qquad (23)$$

Where:

$$x=(-1)^{(m+1)/2}\left[\frac{b\,G_1}{4\tan h(\rho_m)\tan(\theta_m)+\cosh(\rho_m)\cos(\theta_m)}\right]$$

$$\left[C_m\left(\delta_m+\alpha_m\frac{\tan[\theta_m]}{\tan h(\rho_m)}\right)+D_m\left(\frac{\delta_m}{\tan h(\rho_m)}+\frac{\alpha_m}{\tan(\theta_m)}\right)\right]$$

$$y=(-1)^{(m+1)/2}\left[\frac{a\,G_2}{4\tanh(\phi_m)\tan(\Psi_m)+\cosh(\phi_m)\cos(\Psi)_m}\right]$$

$$\left[A_m\left(\beta_m+\gamma_m\frac{\tan\Psi_m}{\tan h\,\Psi_m}\right)+B_m\left(\frac{\beta_m}{\tan h(\Psi_m)}+\frac{\gamma_m}{\tan(\Psi_m)}\right)\right]$$

$$f_{mn}=(-1)^{(n+1)/2}\,(-1)^{(m+1)/2}\left(\frac{\eta_n\,\delta_m}{\bar{\alpha}_m\,\alpha^*_m(\delta_m^2+\bar{\alpha}_m^2)(\delta_n^2+\alpha_m^2)}\right)$$

$$\left\{\frac{1}{\tan\theta_m}-\frac{1}{\tan h\,\rho_m\tan\theta_m}\right\}$$

$$\left[\delta_m^2\left(\alpha_m^2-\mu_n^2\right)+\alpha_m^4-\mu_n^4+\alpha_m\,\delta_m^3+\alpha_m^3\,\delta_m+3\alpha_m\,\delta_m\mu_n^2\right.$$

$$\left.-2\left(\delta_m^2+\bar{\alpha}_m^2\right)\left(\delta_m^2+\delta_m^{*2}\right)\right]+(\tan\rho_m+1)$$

$$\left[2\mu_n\,\alpha_m\left(\alpha_m^2-\mu_n^2+\delta_m\,\mu_n\right)+\delta_m\,\alpha_m\left(\delta_m^2+\alpha_m^2+\mu_n^2\right)\right.$$

$$\left.-\mu_n^2\left(\delta_m^2+\bar{\alpha}_m^2+\alpha_m^{*2}\right)\right]$$

$$e_{mn}=(-1)^{(n+1)/2}\,(-1)^{(m+1)/2}$$

$$\left(\frac{\mu_n\,\beta_m}{\bar{\gamma}_m\,\gamma^*_m\,(\beta_m^2+\bar{\gamma}_m^2)\,(\beta_m^2+\alpha_m^2)}\right)$$

$$\left\{\frac{1}{\tan\Psi_m}-\frac{1}{\tan h\,\phi_m\tan\Psi_m}\right)$$

$$\left[\beta_m^2\,(\gamma_m^2-\eta_n^2)+\gamma_m^4-\eta_n^4+\gamma_m\beta_m^3+\gamma^3_m\beta_m+3\gamma_m\beta_m\eta_n^2\right.$$

$$\left[2(\beta_m^2+\bar{\mu}_m^2)(\beta_m^2+\gamma_m^{*2})\right]+(\tan h\,\phi_m+1)$$

$$\left[2\eta_n\gamma_m(\gamma_m^2-\eta_n^1+\beta_m\eta_n)+\beta_m\gamma_m\,(\beta_m^2+\gamma_m^2+\eta_n^{*2})\right.$$

$$\left.\left.-\eta_n^2\,(\beta_m^2+\bar{\gamma}_m^2)(\beta_m^2+\gamma_m^{*2})\right]\right\}$$

$$e_m=\left\{\frac{\beta_m}{\tan h\,\phi_m}+\frac{\gamma_m}{\tan\Psi_m}-\beta_m\tan h\,\phi_m+\gamma_m\tan\Psi_m\right\}a/4$$

$$f_m=\left\{\frac{\delta_m}{\tan h\,\rho_m}+\frac{\alpha_m}{\tan\theta_m}-\delta_m\tan h\,\rho_m+\alpha_m\tan\theta_m\right\}b/4$$

Arranging equation (21) and (23) in matrix form yields

$$\begin{bmatrix}[e_{mm}] & [f_m]\\ [e_m] & [f_{mn}]\end{bmatrix}\begin{bmatrix}E_m\\ F_m\end{bmatrix}=\begin{bmatrix}x\\ y\end{bmatrix} \qquad (24)$$

where the notation, indicates a diagonal matrix, because of diagonal matrices, a condensation of the matrix can be performed as given in Ref. 4.

Discussion:

In the previous sections a complete solution for the plate resting on elastic foudation is obtained in the form of an infinite series.

A particular solution to specific plate is selected by choosing different boundary conditions to satisfy the equation (B).

If the chosen plate was not symmetrical with respect to the boundary conditions, the even and odd solutions of equation (1) must be included in equation (B).

It has been shown in reference (2 - 5) that this method produced excellent results with a remarkably small amount of computational effort; i.e. the computed values converge with use of a very few terms in each series, thus permitting the use of personal computers.

While the analytic formulation (and computer program development) is not easy using this method.

Acknowledgements:

The author wishes to express gratitude to J.R.Hutchinson for enlightenment on various aspects of his previous work.

Appendix - NOTATION

The following symbols are used in this paper:

a = The plate width in x−direction

b = The plate width in y-direction

D $= \dfrac{Eh^2}{12(1-\nu^2)}$ «plate rigidity»

E = Modulus of elasticity

h = The plate thickness

ν = Poisson's ratio

W = The plate deflection

q = Intensity of the lateral load

$\mu_m = \dfrac{m\pi}{a}$

$$\mu_m = \frac{m\pi}{b}$$

$$2\beta_m^2 = \sqrt{\mu_m^4 + \frac{K}{D}} + \mu_m^2$$

$$2\gamma_m^2 = \sqrt{\mu_m^4 + \frac{K}{D}} - \mu_m^2$$

$$2\delta_m^2 = \sqrt{\mu_m^4 + \frac{K}{D}} + \mu_m^2$$

$$2\alpha_m^2 = \sqrt{\mu_m^4 + \frac{K}{D}} - \mu_m^2$$

$$\theta_m = \frac{\alpha_{ma}}{2}$$

$$\rho_m = \frac{\delta_{ma}}{2}$$

$$\Psi_m = \frac{\gamma_{mb}}{2}$$

$$\theta_m = \frac{\beta_{mb}}{2}$$

$$\bar{\gamma}_m = \gamma_m - \mu_n$$

$$\gamma_m^* = \gamma_m + \mu_n$$

$$\bar{\alpha}_m = \alpha_m - \mu_n$$

$$\alpha_m^* = \alpha_m + \mu_n$$

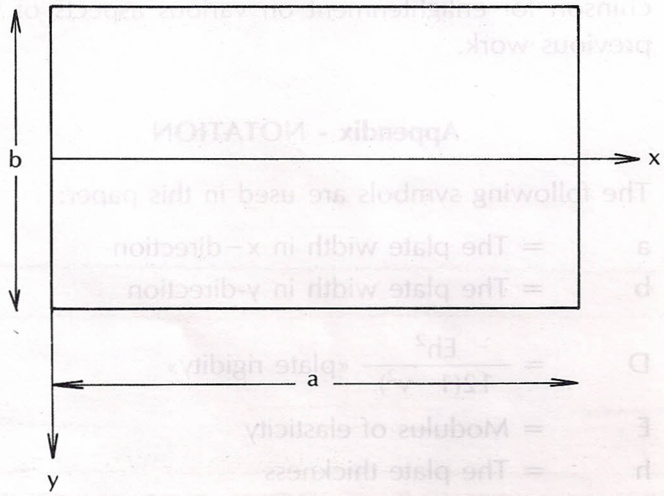

Fig.1

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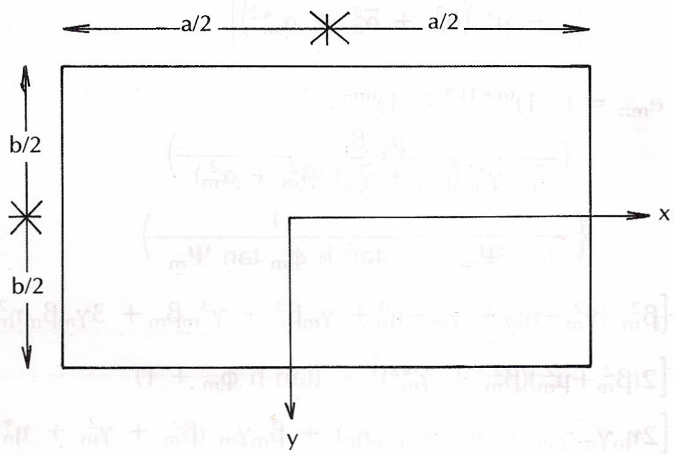

Fig.2